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Mathematics. — *On the geometry of spin spaces.* I. By J. A. SCHOUTEN.

(Communicated at the meeting of May 28, 1949.)

§ 1. *Object transformations and coordinate transformations in an R_n and in its spin space.*

In the foundation of every theory of spin spaces we find in some form or other the following formula

$$v^A_{\cdot B} = \eta^A_{\cdot B\varkappa}\, v^\varkappa ; \qquad A, B = 1, \ldots, N; {}^{1)} \quad . \quad . \quad . \quad . \quad (1.1)$$

where the $v^\varkappa$ are the components of a contravariant vector in a centred R_n and the $v^A_{\cdot B}$ are components of a mixed quantity of valence 2 in a centred E_N, both with respect to rectilinear affine coordinatesystems $(\varkappa)$ and (A) respectively [2]. Now (1.1) expresses that among the mixed affinors of valence 2 in E_N there is a certain set of ∞^n that is looked upon as vectors in an R_n. But we need something more to get at the conception of spin space. This may be formulated for $n = 2\nu$ [3]. In R_n we have to introduce a one to one correspondence between a set of mutually perpendicular unit-vectors $\underset{j}{i}{}^\varkappa$; $j = 1, \ldots$, n with signatures $\underset{j}{\varepsilon}{}^2$; $\underset{j}{\varepsilon} = 1$ or i, and a set of quantities $\underset{j}{i}{}^A_{\cdot B}$ in E_N; $N = 2^\nu$, such that

$$\underset{(j}{i}{}^A_{\cdot C}\, \underset{k)}{i}{}^C_{\cdot B} = \begin{cases} \mathfrak{A}^A_B \text{ for } j = k \\ 0 \quad \text{for } j \neq k \end{cases}; \; \mathfrak{A}^A_B = \begin{cases} 1 \text{ for } A = B \\ 0 \text{ for } A \neq B \end{cases}. \quad . \quad . \quad (1.2)$$

$\eta^A_{\cdot B\lambda}$ is a connecting quantity, connecting the R_n and the E_N and transforming as follows

$$\eta^{A'}_{\cdot B'\lambda'} = \mathfrak{A}^{A'B}_{A\,B'}\, A^\lambda_{\lambda'}\, \eta^A_{\cdot B\lambda} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

if in both spaces homogeneous linear coordinate transformations are performed. From our point of view it has no sense to ask for a geometric interpretation of a change of the quantity η. Points and other objects in the E_n are really objects of R_n, and the only meaning of the introduction of the E_N is that these objects get other components that transform in a convenient way. Hence the quantity η has only the task of furnishing these other components. It behaves like the unity affinor $A^\varkappa_\lambda$ in R_n that furnishes new coordinates for a vector in the formula $v^{\varkappa'} = A^{\varkappa'}_\varkappa\, v^\varkappa$.

It has no sense to ask for a change of the unity affinor, only its mixed components $A^{\varkappa'}_\varkappa$ may change with coordinate transformations, and the same holds for the quantity η.

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1) If no special agreement is made, the indices $\varkappa, \lambda, \ldots$ always take the values $1, \ldots, n$.

2) An E_n is an ordinary affine space, a centred E_n is an E_n with fixed origin. An R_n is an E_n with a metric fixed by a fundamental tensor with constant components.

3) The case $n = 2\nu + 1$ can be easily dealt with if we know all about the case $n = 2\nu$.

The most important theorem of spin theory states that if there is given an *orthogonal vector*transformation

$$\underset{j'}{i}^{\varkappa} = T^{\varkappa}_{.\lambda} \underset{j}{i}^{\varkappa} \quad \ldots\ldots\ldots \quad (1.4)$$

in R_n, there always exists an *affine vector*transformation $S^A_{.B}$ in the E_N, such that

$$\underset{j'}{i}^{A}_{.B} = S^A_{.C} \underset{j}{i}^C_{.D} \overset{-1}{S}{}^D_{.B} \quad \ldots\ldots\ldots \quad (1.5)$$

From (1.1), (1.4) and (1.5) it follows, that the quantity η is invariant if both *object*transformations are performed simultaneously

$$\eta^A_{.B\lambda} = \eta^C_{.D\varkappa} S^A_{.C} \overset{-1}{S}{}^D_{.B} \overset{-1}{T}{}^{\varkappa}_{.\lambda} \quad \ldots\ldots\ldots \quad (1.6)$$

This same theorem, formulated in terms of coordinate transformations, states that if $(h) \to (h')$ is an *orthogonal coordinate* transformation in R_n, there always exists an *affine coordinate* transformation $(A) \to (A')$ in $\mathfrak{S}_N$ such that the *components* of $\eta^A_{.Bi}$ are individually invariant:

$$\eta^{A'}_{.B'i'} = \delta^{A'}_{A}\, \delta^{B}_{B'}\, \delta^{i}_{i'}\, \eta^A_{.Bi}; \quad \ldots\ldots\ldots \quad (1.7)$$

$\delta =$ Kronecker symbol.

That means, that, if we choose orthogonal coordinatesystems (h) in R_n; $h = 1, \ldots, n$, the coordinatesystems in R_n and E_N can always be linked in such a way, that the components $\eta^A_{.Bi}$ have during the investigation always the same definitely chosen values. This is often done but it will not be done hereafter. In contradistinction to many other publications we leave the coordinatesystems in E_N as long as possible entirely free as far as they cannot be fixed by really invariant conditions.

The correspondence $T \to S$ furnishes a *representation* of the orthogonal group in R_n. Here arises the problem of the determination of all independent representations of this group and its main subgroups, a problem that is dealt with thoroughly in many publications and that will not be touched hereafter. The problem we are interested in here is another one. To every orthogonal transformation in R_n there correspond at least two homogeneous linear transformations in E_N and these form a group. It is this group that we intend to investigate. Especially we face the problem how this group can be determined by its invariants [4]) or by some invariants and some conditions. Some of these invariants, the quantities named hereafter $R^A_{.B}$, C_{AB}, $\Omega_{\bar{A}B}$ and $\Pi^{\bar{A}}_{.B}$ and called the *chief invariants*, are well known. But for $\nu > 2$ these chief invariants do not determine the group and accordingly in these cases there must be more invariants. We prove that there is an invariant $S^{A\,\cdot\,C}_{.B\,.D}$ that determines the whole group and together with $\Omega_{\bar{A}B}$

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[4]) We use the term invariant in its broader sense, including invariant quantities, not only scalars.

the group corresponding to all *real* orthogonal transformations in R_n. For $\nu = 3$ there is another very simple invariant of valence four determining the whole group together with R and C and the subgroup corresponding to *real* transformations together with R, C and Ω. For $\nu > 3$ it is more convenient to complete R, C and Ω by adding some conditions that will turn out to be very simple.

As to former publications we mention here only the most important ones of VAN DER WAERDEN [5]), BRAUER and WEYL [6]), VEBLEN and GIVENS [7]) and CARTAN [8]). VAN DER WAERDEN's paper gave the foundation of spinor analysis for $\nu = 2$. His point of view is the same as ours. BRAUER and WEYL were most interested in representation theory, they fix from the beginning a certain representation, that is a certain coordinate system in E_N where we try to leave this coordinate system as free as possible. In CARTAN's two little books the coordinatesystems in E_N are also fixed but he is chiefly interested in the geometry in E_N, just as we are. VEBLEN and GIVENS consider the auxiliary space from the projective point of view and make plenty use of the wellknown properties of collineations, anticollineations, correlations and anticorrelations in projective space. Many points of our exposition are closely connected with points of these lectures [9]). But there are two big differences, firstly we take always the affine point of view and secondly we use only very elementary means and do not need any more complicated geometrical results. Other publications will be referred to hereafter.

§ 2. *Associative algebras.* We recall briefly some properties of hypercomplex number systems necessary in the sequel [10]). If m hypercomplex numbers $e_1, \ldots, e_m$ form an algebra with the multiplication rules

$$e_\mu e_\lambda = c_{\mu\lambda}^{\cdot\cdot\varkappa} e_\varkappa \quad ; \quad \varkappa, \lambda, \mu = 1, \ldots, m \quad . \quad . \quad . \quad . \quad (2.1)$$

with m^3 ordinary real or complex numbers $c_{\mu\lambda}^{\cdot\cdot\varkappa}$ this algebra is associative if and only if

$$c_{\mu\lambda}^{\cdot\cdot\varrho} c_{\varrho\nu}^{\cdot\cdot\varkappa} = c_{\mu\varrho}^{\cdot\cdot\varkappa} c_{\lambda\nu}^{\cdot\cdot\varrho} \quad ; \quad \varkappa, \lambda, \mu, \nu, \varrho = 1, \ldots, m.. \quad . \quad . \quad (2.2)$$

If the e_λ suffer a homogeneous linear transformation, the $c_{\lambda\mu}^{\cdot\cdot\mu}$ transform in the same way as the e_λ. For each value of λ the $c_{\lambda\mu}^{\cdot\cdot\mu}$ is called the *trace* (spur, Spur) of the hypercomplex number e_λ. Each set of m linearly independent hypercomplex numbers is a *base* of the algebra. We only consider associative algebras containing a number A satisfying the condition

$$AB = BA = B \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

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5) Spinoranalyse, Nachr. d. Ges. d. Wiss. Göttingen 1929, 100—109.

6) Spinors in n dimensions, Amer. Journ. of Math. 57, 425—449 (1935).

7) Geometry of complex domains, Lectures Inst. of adv. Study 1935—'36.

8) Leçons sur la théorie des spineurs I and II, Act. scient. et industr. 643 and 701, 1938.

9) It is a pity that these beautiful lectures are not printed and only known to a few persons that have got a cyclostyled copy.

10) Cf. e.g. L. E. DICKSON, Algebras and their Arithmetics, Chicago 1923.

for every choice of the hypercomplex number B. A is called *unity* and is mostly identified with 1. The trace of A is m.

The mixed affinors $P^{A}_{.B}$; $A, B = 1, \ldots, N$ in an E_N with the multiplication rules

$$P^{A}_{.B}\, Q^{B}_{.C} = R^{A}_{.C}\ ;\ \text{short}: PQ = R;\quad A, B, C = 1, \ldots, N. \quad (2.4)$$

form an associative algebra with $m = N^2$, called the *matrix algebra* M_N. Obviously the trace of $P^{A}_{.B}$ is $NP^{A}_{.A}$. It has been proved that *every associative algebra is a subalgebra of some* M_N.

By multiplication of two algebras with bases e_λ; $\lambda = 1, \ldots, m$ and $\mathbf{e}_\Lambda$; $\Lambda = 1, \ldots, M$ an algebra with mM numbers $e_\lambda \times \mathbf{e}_\Lambda = \mathbf{e}_\Lambda \times e_\lambda$ can be formed with the multiplication rules

$$(e_\lambda \times \mathbf{e}_\Lambda)(e_\mu \times \mathbf{e}_M) = (e_\lambda e_\mu) \times (\mathbf{e}_\Lambda \mathbf{e}_M). \quad \ldots \ldots \quad (2.5)$$

The product of $s\,M_2$'s is an M_{2s} and conversely M_N can for N even be written as the product of $\frac{1}{2}N\, M_2$'s in an infinite number of ways.

A hypercomplex number F for which $FF = F$ is called an *idempotent*. To every associative algebra there exists an arithmetic invariant p indicating the maximum number of idempotents whose mutual products are zero. Such a set of p idempotents is called a *main set* (Hauptreihe) of the algebra. E.g. in the M_N of the $P^{A}_{.B}$ we have $p = N$ and the N affinors

$$\underset{1}{e}^{A} \overset{1}{e}_B; \ldots\ ;\ \underset{N}{e}^{A} \overset{N}{e}_B\ ;\ A, B = 1, \ldots, N \quad \ldots \quad (2.6)$$

formed by the measuring vectors of the E_N form a main set. For main sets the following theorem holds:

Theorem I. *(Theorem of autoisomorphism of associative algebras.) If the base $e_1, \ldots, e_m$ is chosen in such a way that $e_1, \ldots, e_p$ form a main set and if $E_1, \ldots, E_p$ is another main set, there exist always $m - p$ hypercomplex numbers $E_{p+1}, \ldots, E_m$, such that the multiplication rules of the e_λ and of the E_λ; $\lambda = 1, \ldots, m$, are the same.*

This theorem was proved by the author in 1914 [11]). In the proof the E_λ were expressed explicitly in the e_λ [12]) and from these expressions it follows immediately that in the special case that the algebra is an M_N there exists a hypercomplex number S of M_N such that

$$E_\lambda = S e_\lambda S^{-1} \quad ; \quad \lambda = 1, \ldots, m \quad \ldots \ldots \quad (2.7)$$

and that this number is determined to within a real or complex scalar factor. But this conclusion was not drawn in 1914 and it was formulated and proved independently by BRAUER and WEYL in 1935 [13]). Calling a

11) Zur Klassifizierung des assoziativen Zahlensysteme, Math. Ann. **76**, 1—66 (1914).
12) L.c. p. 34.
13) L.c. p. 447.

transformation of the form $S \dots S^{-1}$ (that obviously leaves invariant the multiplication rules of the algebra) an *automorphism*, the theorem of BRAUER and WEYL can be formulated as follows:

Theorem II. *(Theorem of automorphisms of an M_N.)*

If $e_1, \dots, e_{N^2}$ and $E_1, \dots, E_{N^2}$ are two bases of an M_N with the same multiplication rules, there always exist an automorphism

$$E_\lambda = S e_\lambda S^{-1} \quad ; \quad \lambda = 1, \dots, N^2 \quad . \quad . \quad . \quad . \quad (2.8)$$

in which the hypercomplex number S is determined to within a real or complex scalar factor.

A *Clifford algebra* C_n is an associative algebra consisting of 2^n numbers, viz. A (or 1), n numbers $\underset{1}{i}, \dots, \underset{n}{i}$, satisfying the multiplication rules

$$\underset{j}{i}\,\underset{k}{i} = \begin{cases} \underset{j}{\varepsilon}^2 \text{ for } j = k & ; \quad \underset{j}{\varepsilon}^2 = \pm 1 \\ -\underset{k}{i}\,\underset{j}{i} \text{ for } j \neq k & ; \quad j, k = 1, \dots, n \end{cases} \quad . \quad . \quad . \quad (2.9)$$

and all products of the $\underset{j}{i}$'s up to $I \overset{\text{def}}{=} \underset{1}{i} \dots \underset{n}{i}$. This set of 2^n numbers is called a *Clifford base* of the algebra. The set of numbers $\underset{1}{i}, \dots, \underset{n}{i}$ is called a *Clifford set* and the sequence $\underset{1}{\varepsilon}^2, \dots, \underset{n}{\varepsilon}^2$ its *signature*. It has been proved that for $n = 2\nu$ a C_n is an $M_N; N = 2^\nu$ and that for $n = 2\nu + 1$ a C_n is the product of an $M_N; N = 2^\nu$ and a C_1 with two numbers A (or 1) and B and the multiplication rules

$$AA = A \quad ; \quad AB = BA = A \quad ; \quad BB = A \quad . \quad . \quad (2.10)$$

From this it follows, making use of theorem II, that for $n = 2\nu$ every set of n numbers of a C_n, satisfying equations of the form (2.9), is a Clifford set, i.e. that these numbers and their products form a Clifford base. From the definition of the trace it follows that *all numbers of a Clifford base except A have trace zero and that the trace of A is* 2^n.

Using the fact that $C_n = M_N$; $N = 2^\nu$ for $n = 2\nu$ we may write one of the numbers of a Clifford set, e.g. $\underset{1}{i}$ as a matrix $\underset{1}{i}{}^A{}_B$; $A, B = 1, \dots, N$. Then, if v^A is an eigenvalue of $\underset{1}{i}$, we have

$$\underset{1}{i}\, v = \lambda v \quad ; \quad \underset{1}{i}\,\underset{1}{i}\, v = \underset{1}{\varepsilon}^2 v = \lambda^2 v \quad . \quad . \quad . \quad . \quad . \quad (2.11)$$

and consequently $\lambda = \pm \underset{1}{\varepsilon}$. The trace of $\underset{1}{i}$ being zero, $\underset{1}{i}$ must have just $M \overset{\text{def}}{=} \frac{1}{2} N$ eigenvalues $+\underset{1}{\varepsilon}$ and M eigenvalues $-\underset{1}{\varepsilon}$. In the same way it can be proved that the number $\underset{1}{i} \dots \underset{p}{i}$; $p \leq n$ has M eigenvalues $+ i^{(p_2)} \underset{1}{\varepsilon} \dots \underset{p}{\varepsilon}$ and M eigenvalues $- i^{(p_2)} \underset{1}{\varepsilon} \dots \underset{p}{\varepsilon}$. From this it follows geometrically that *every*

number of a Clifford base except A fixes in E_N a set of two E_M's having no direction in common.

We prove that *in a $C_{2\nu+\varepsilon}$; $\varepsilon = 0$ or 1 there exist sets of $2\nu + 1$ and no sets of more numbers that are at the same time square roots of A or $-A$ and mutually anticommutative.* We take any Clifford set $\underset{1}{i}, \ldots, \underset{n}{i}$. Be now R a square root of A, anticommutative with each of the i's. Then, if R is written out in A, the i's and the products of the i's up to I, we see that in this expression all terms vanish for $\varepsilon = 1$ and that only the term with I remains for $\varepsilon = 0$. Hence for $\varepsilon = 0$

$$R = \pm i^{\nu} \underset{1}{\varepsilon} \ldots \underset{n}{\varepsilon} I. \qquad (2.12)$$

For $\varepsilon = 1$ the same formula (2.12) defines the only number in $C_{2\nu+1}$ that is at the same time 1° *not equal to A*, 2° *a square root of A* and 3° *commutative with all numbers of $C_{2\nu+1}$*. Obviously for $\varepsilon = 1$ the number R is invariant to within the sign for all autoisomorphisms. But this is not true for $\varepsilon = 0$; e.g. for $\nu = 1$ the transformation

$$\underset{1'}{i} = \underset{1}{i} \quad ; \quad \underset{2'}{i} = i \varepsilon \underset{1}{i}\underset{1}{i}\underset{2}{i} \quad ; \quad \underset{1'}{i}\underset{2'}{i} = i \varepsilon^3 \underset{1}{i}\underset{2}{i} \qquad (2.13)$$

is an automorphism.

In a $C_{2\nu}$ a Clifford set $\underset{j}{i}$ being given, the $1 + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n} = {}^1/_2 N^2$ products of an even number of $\underset{j}{i}$'s form a $C_{2\nu-1}$, which is a product of $\nu - 1$ M_2's and the C_1 formed by A and the R belonging to this Clifford set. In this $C_{2\nu-1}$ not only A but also R is commutative with all other numbers. A Clifford set of this algebra is e.g.

$$\underset{1}{i}\underset{2}{i} \; ; \; \underset{1}{i}\underset{3}{i} \; ; \; \ldots \; ; \; \underset{1}{i}\underset{2\nu}{i}. \qquad (2.14)$$

§ 3. *The spin space belonging to an R_n: $n = 2\nu$.* Be $\underset{j}{i}^{\varkappa}$; $j = 1, \ldots, n$, $\varkappa = 1, \ldots, n$ a set of n mutually orthogonal unitvectors in an R_n with signature $\underset{1}{\varepsilon}^2, \ldots, \underset{n}{\varepsilon}^2$. From these vectors $\binom{n}{p}$ p-vectors $\underset{j_1}{i}^{[\varkappa_1}\ldots\underset{j_p}{i}^{\varkappa_p]}$ can be formed for every value of p from 2 up to n. Together with the scalar 1 we have then 2^n quantities. For two quantities of this set, containing p and q vectorfactors respectively and having just r vectorfactors in common we define a product as follows. After writing the quantities in the right order, the transvection with $g_{\lambda\varkappa}$ is formed over the two neighbouring indices, then over the next two neighbouring indices, and so on r times, and from the remaining factors the alternating product is formed multiplied by $\binom{p}{r}\binom{q}{r} r!$, e.g.

$$\underset{1}{i}^{[\varkappa}\underset{2}{i}^{\lambda}\underset{3}{i}^{\mu]}; \; \underset{4}{i}^{[\nu}\underset{2}{i}^{\varrho}\underset{3}{i}^{\tau]} \rightarrow \underset{1}{i}^{[\varkappa}\underset{2}{i}^{\lambda}\underset{3}{i}^{\mu]}\underset{4}{i}^{[\nu}\underset{2}{i}^{\varrho}\underset{3}{i}^{\tau]} g_{\mu\nu} g_{\lambda\varrho} \rightarrow -\underset{2}{\varepsilon}^2\underset{3}{\varepsilon}^2 \underset{1}{i}^{[\varkappa}\underset{4}{i}^{\tau]}. \qquad (3.1)$$

This being done, the 2^n quantities with this multiplication form a C_n. Writing the unitvectors simply $\underset{j}{i}$; $j = 1, \ldots, n$ and denoting their alternating products by $\underset{j}{i}\underset{k}{i}$ or $\underset{jk}{i}$ etc., the $\underset{j}{i}$'s form a Clifford set with the multiplication rules (2.9).

The C_n being an $M_N; N = 2^\nu$, the i's and their products $\underset{jk}{i}, \underset{jkl}{i}, \ldots$ may be considered as affinors in an auxiliary E_N

$$\left.\begin{aligned} \underset{j}{i}{}^{.A}_{.B} &= \eta^{A}_{.B\varkappa}\, \underset{j}{i}{}^{\varkappa} \quad ; \quad A, B = 1, \ldots, N \\ \underset{jk}{i}{}^{.A}_{.B} &= \underset{j}{i}{}^{.A}_{.C}\, \underset{k}{i}{}^{.B}_{.D} \qquad\qquad\qquad \text{etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3.2)$$

and vice versa every mixed affinor $P^{A}_{.B}$ of valence 2 in E_N may be considered as a definite set of quantities in R_n consisting of a scalar, a vector, a bivector etc. up to an n-vector. More general every object, e.g. a vector (or point) in E_N may be considered as a definite geometric object in R_n consisting of a set of quantities in R_n. The E_N connected in this way with the R_n is called the *spinspace* of R_n and from now on denoted by $\mathfrak{S}_N$; $N = 2^\nu$.

If a Clifford set $\underset{j}{i}$ of C_n is given and another Clifford set $\underset{j'}{i}$ *with the same signature*, according to theorem II there exists an automorphism $S \ldots S^{-1}$ such that

$$\underset{j'}{i}{}^{A}_{.B} = S^{A}_{.C}\, \underset{j}{i}{}^{C}_{.D}\, \overset{-1}{S}{}^{D}_{.B} \quad ; \quad \text{short: } \underset{j'}{i} = S \underset{j}{i} S^{-1}. \quad . \quad . \quad . \quad (3.5)$$

$S^{A}_{.B}$ represents a homogeneous linear transformation of vectors in $\mathfrak{S}_N$ and is determined to within a scalar factor. Conversely every linear homogeneous transformation in $\mathfrak{S}_N$ transforms every base of C_n into another base with the same multiplication rules.

The $\underset{j'}{i}$'s in (3.5) can always be linearly expressed in the $\underset{j}{i}$'s and their products. But it may happen that in these expressions only the $\underset{j}{i}$'s occur and *not* their products. Then we have in $\mathfrak{S}_N$ a homogeneous linear transformation transforming the $\underset{j}{i}$'s into linear forms of the $\underset{j}{i}$'s and leaving all scalar products invariant. But this corresponds to an orthogonal transformation in R_n.

$$\underset{j'}{i}{}^{\varkappa} = T^{\varkappa}_{.\lambda}\, \underset{j}{i}{}^{\varkappa}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

Conversely every orthogonal transformation T in R_n corresponds to such a special transformation S in $\mathfrak{S}_N$ and this latter transformation is fixed to within an arbitrary real or complex scalar factor. By the condition $|\,\mathrm{Det}\ (S)| = +1$, S can be determined to within a scalar factor of the form $e^{i\varphi}$. Hence the group G_{or} of all orthogonal transformations in R_n

corresponds to a certain subgroup G_s of the group of all affine transformations in $\mathfrak{S}_N$. Every correspondence between the transformations of G_{or} and some linear homogeneous group in another space is called a *representation of* G_{or}. As has been established in the well developed theory of representations the representation in $\mathfrak{S}_N$ considered here is only a special case and other representations can be derived from it, e.g. by considering the transformations in $\mathfrak{S}_N$ of other quantities instead of vectors. But this is not what we are interested in here. Our chief aim is to study the group G_s and its comitants.

The equations (2.9) can be written

$$\underset{(j}{i}{}^{A}{}_{.C}\underset{k)}{i}{}^{C}{}_{.B} = g_{jk}\,\mathfrak{A}^A_B \quad ; \quad \begin{matrix} j,k=1,\ldots,n; \\ A,B,C=1,\ldots,N \end{matrix} \quad \ldots \quad (3.7)$$

and in this formula we recognize for $n=4$ the $\underset{j}{i}{}^{A}{}_{.B}$ as the wellknown hypercomplex numbers or matrices introduced by DIRAC for $\nu=2$.

(3.2) and (3.5) establish a definite correspondence between vectortransformations of G_{or} in R_n and vectortransformations of G_s in $\mathfrak{S}_N$. If in $\mathfrak{S}_N$ we perform the coordinatetransformation $(A)\rightarrow(A')$ this same correspondence will be represented by other components $S^{A'}_{.B'}$. So every definite choice of a coordinatesystem in $\mathfrak{S}_N$ gives another representation of G_{or}. As to these coordinatesystems we try to fix them as far as possible by invariant conditions. Special coordinatesystems in $\mathfrak{S}_N$ not fixed in an invariant way will only be used temporarily for computing purposes.

By the orthogonal coordinatesystems in R_n a certain kind of Clifford sets in the C_n is preferred and with them the quantity R, known to within the sign. Hence this quantity is invariant for all transformations in $\mathfrak{S}_N$ corresponding to rotations in R_n and it changes its sign for all transformations corresponding to reflexotations [14]). Now we have seen that R fixes two E_M's in $\mathfrak{S}_N$ and that enables us to introduce as a *first invariant condition* that from the contravariant measuring vectors in $\mathfrak{S}_N$ the first M lie in one invariant E_M and the other ones in the other invariant E_M. Then R takes the form

$$R^A_{.B} = \pm(\underset{1}{e}{}^A \overset{1}{e}_B + \ldots + \underset{M}{e}{}^A \overset{M}{e}_B - \underset{M+1}{e}{}^A \overset{M+1}{e}_B - \ldots - \underset{N}{e}{}^A \overset{N}{e}_B) \ldots \quad (3.8)$$

For $\nu=1$ we have $n=2$, $N=2$ and the M_2 is a quaternion algebra. Denoting R in this case by r, it is wellknown that there exist in M_2 next to r two hypercomplex numbers p and q such that

$$\left.\begin{matrix} pq=-ir & ; & pp=1 \\ qr=-ip & ; & qq=1 \\ rp=-iq & ; & rr=1 \end{matrix}\right\} \ldots\ldots \quad (3.9)$$

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[14]) A reflexotation in R_n is a transformation with determinant -1. It can be formed by a rotation (det. $=+1$) followed by a reflexion at an R_{n-1}.

and that, the representation of r being given by (3. 8), the only possible representations of p and q are

$$\left.\begin{array}{lcl} p=\lambda\, \underset{1}{e}{}^{A}\, \overset{2}{e}_{B} + {}^{1}/_{\lambda}\, \underset{2}{e}{}^{A}\, \overset{1}{e}_{B} & & p=\lambda\, {}^{1}{}_{2} + {}^{1}/_{\lambda}\, {}^{2}{}_{1} \\ & ;\ \text{short:} & \\ q=i\lambda\, \underset{1}{e}{}^{A}\, \overset{2}{e}_{B} - i\, {}^{1}/_{\lambda}\, \underset{2}{e}{}^{A}\, \overset{1}{e}_{B} & & q=i\lambda\, {}^{1}{}_{2} - i\, {}^{1}/_{\lambda}\, {}^{2}{}_{1} \end{array}\right\} \quad . \quad (3.\,10)$$

with an *arbitrary real or complex* parameter λ. Hence for $\nu=1$ the most general representation, satisfying the first invariant condition is

$$\left.\begin{array}{r} \underset{1}{i}=\underset{1}{\varepsilon}\, p=\underset{1}{\varepsilon}\,(\lambda\, {}^{1}{}_{2}+{}^{1}/_{\lambda}\, {}^{2}{}_{1}) \\ \underset{2}{i}=\underset{2}{\varepsilon}\, q=i\,\underset{2}{\varepsilon}\,(\lambda\, {}^{1}{}_{2}-{}^{1}/_{\lambda}\, {}^{2}{}_{1}) \\ I=\underset{1}{i}\,\underset{2}{i}=-i\,\underset{1}{\varepsilon}\,\underset{2}{\varepsilon}\, r=-i\,\underset{1}{\varepsilon}\,\underset{2}{\varepsilon}\,({}^{1}{}_{1}-{}^{2}{}_{2}) \end{array}\right\} \quad . \ . \ . \ . \quad (3.\,11)$$

from which we see that *real* representations are only possible for the two indefinite cases: $\lambda=$ real for $\underset{1}{\varepsilon}=1$, $\underset{2}{\varepsilon}=i$ and $\lambda=$ imaginary for $\underset{1}{\varepsilon}=i$, $\underset{2}{\varepsilon}=1$.

For $\nu=2$ we have $n=4$, $N=4$ and the M_4 can be written as the product of two M_2's. We have already proved that there exist sets of 5 (and no sets of more) numbers that are square roots of A and mutually anti-commutative. EDDINGTON called them *pentads*. Writing out the 16 products of the numbers $1, \overset{1}{p}, \overset{1}{q}, \overset{1}{r}$ and $1, \overset{2}{p}, \overset{2}{q}, \overset{2}{r}$ of the two M_2's and omitting the indices because in the sequel the order of two of these factors will never be changed, we get the 16 numbers of M_4

$$\left.\begin{array}{cccc} 1\times 1 & 1\times p & 1\times q & 1\times r \\ p\times 1 & p\times p & p\times q & p\times r \\ q\times 1 & q\times p & q\times q & q\times r \\ r\times 1 & r\times p & r\times q & r\times r \end{array}\right\} \quad . \ . \quad (3.\,12)$$

Now it is easy to verify that the 5 numbers

$$p\times 1 \ ;\ r\times p \ ;\ q\times 1 \ ;\ r\times q \ ;\ r\times r \quad . \ . \quad (3.\,13)$$

form a pentad and that every other pentad taken from the 16 numbers (3. 12) can be obtained from (3. 13) by suitable permutations of $\overset{1}{p}, \overset{1}{q}, \overset{1}{r}$ and $\overset{2}{p}, \overset{2}{q}, \overset{2}{r}$. From theorem II it follows moreover that, an arbitrary pentad in M_4 being given, it is always possible to find in M_4 two M_2's and in each M_2 four numbers $1, p, q, r$ such that the given pentad is just represented by (3. 13).

From the pentad (3. 13) four numbers can be taken only in two essentially different ways, i.e. we may leave out either a product containing

a symbol 1 or a product without a 1. Hence the two different possibilities for the representation of $\underset{1}{i}, \ldots, \underset{4}{i}$ are

$$\left.\begin{array}{lcc}
 & \mathrm{I} & \mathrm{II} \\
\underset{1}{i}: & \underset{1}{\varepsilon}\, p\times 1 & \underset{1}{\varepsilon}\, r\times p \\
\underset{2}{i}: & \underset{2}{\varepsilon}\, r\times p & \underset{2}{\varepsilon}\, r\times q \\
\underset{3}{i}: & \underset{3}{\varepsilon}\, q\times 1 & \underset{3}{\varepsilon}\, r\times r \\
\underset{4}{i}: & \underset{4}{\varepsilon}\, r\times q & \underset{4}{\varepsilon}\, p\times 1 \\
I: & \underset{1234}{\varepsilon\varepsilon\varepsilon\varepsilon}\, r\times r & -\underset{1234}{\varepsilon\varepsilon\varepsilon\varepsilon}\, q\times 1 \\
R: & \pm\, r\times r & \pm\, q\times 1
\end{array}\right\} \quad \ldots \quad (3.14)$$

All other possibilities can be got by permutation of p, q and r in each of the M_2's, by permutation of $\underset{1}{i}, \underset{2}{i}, \underset{3}{i}, \underset{4}{i}$ and by changing the signs of these unitvectors.

If we choose now in each M_2 the representation (3.10) with arbitrary parameters $\underset{1}{\lambda}$ and $\underset{2}{\lambda}$, only I gives a representation of R satisfying our first condition of invariance. For $\underset{1}{\lambda} = 1$, $\underset{2}{\lambda} = 1$, I is the representation used by BRAUER and WEYL [15]) and II is the representation used by KRAMERS to within a sign in $\underset{2}{i}$ (due to a change of sign in the definition of q) [16]).

If the preferred position of r in both M_2's is taken into account we get a third and fourth essentially different possibility

$$\left.\begin{array}{lcc}
 & \mathrm{III} & \mathrm{IV} \\
\underset{1}{i}: & \underset{1}{\varepsilon}\, p\times 1 & \underset{1}{\varepsilon}\, q\times 1 \\
\underset{2}{i}: & \underset{2}{\varepsilon}\, q\times 1 & \underset{2}{\varepsilon}\, p\times q \\
\underset{3}{i}: & \underset{3}{\varepsilon}\, r\times q & \underset{3}{\varepsilon}\, r\times 1 \\
\underset{4}{i}: & \underset{4}{\varepsilon}\, r\times r & \underset{4}{\varepsilon}\, p\times r \\
I: & -\underset{1234}{\varepsilon\varepsilon\varepsilon\varepsilon}\, r\times p & \underset{1234}{\varepsilon\varepsilon\varepsilon\varepsilon}\, p\times p \\
R: & \pm\, r\times p & \pm\, p\times p
\end{array}\right\} \quad \ldots \quad (3.15)$$

As soon as the parameters $\underset{1}{\lambda}$ and $\underset{2}{\lambda}$ are fixed the question arises whether among these representations there are real or purely imaginary ones. We need only to suppose that e.g. both parameters are real (a change of a parameter from real to purely imaginary is equivalent to the interchanging of p and q). Then in both M_2's p and r are real and q is purely imaginary. We then have seven essentially different representations

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[15]) L.c. p. 429.

[16]) Die Grundlagen der Quantentheorie, Leipzig 1938, p. 285, Cf. BELINFANTE, Theory of heavy Quanta, Dissertation, Leiden 1939, p. 7.

$$\left.\begin{array}{l}
\begin{array}{l|r|c||r|c||r|c||}
 & \text{I} & & \text{II} & & \text{II}' & \\
i_1: & \varepsilon_1\, p\times 1 & (r) & \varepsilon_1\, r\times p & (r) & \varepsilon_1\, r\times q & (i) \\
i_2: & \varepsilon_2\, r\times p & (r) & \varepsilon_2\, r\times q & (i) & \varepsilon_2\, r\times p & (r) \\
i_3: & \varepsilon_3\, q\times 1 & (i) & \varepsilon_3\, r\times r & (r) & \varepsilon_3\, r\times r & (r) \\
i_4: & \varepsilon_4\, r\times q & (i) & \varepsilon_4\, p\times 1 & (r) & \varepsilon_4\, q\times 1 & (i) \\
I: & \varepsilon_1 .. \varepsilon_4\, r\times r & & -\varepsilon_1 .. \varepsilon_4\, q\times 1 & & -\varepsilon_1 .. \varepsilon_4\, p\times 1 & \\
R: & \pm\, r\times r & (r) & \pm\, q\times 1 & (i) & \pm\, p\times 1 & (r)
\end{array} \\[2ex]
\begin{array}{l|r|c||r|c||r|c||r|c||}
 & \text{III} & & \text{III}' & & \text{IV} & & \text{IV}' & \\
i_1: & \varepsilon_1\, p\times 1 & (r) & \varepsilon_1\, q\times 1 & (i) & \varepsilon_1\, q\times 1 & (i) & \varepsilon_1\, p\times 1 & (r) \\
i_2: & \varepsilon_2\, q\times 1 & (i) & \varepsilon_2\, p\times 1 & (r) & \varepsilon_2\, p\times q & (i) & \varepsilon_2\, q\times p & (i) \\
i_3: & \varepsilon_3\, r\times q & (i) & \varepsilon_3\, r\times p & (r) & \varepsilon_3\, r\times 1 & (r) & \varepsilon_3\, r\times 1 & (r) \\
i_4: & \varepsilon_4\, r\times r & (r) & \varepsilon_4\, r\times r & (r) & \varepsilon_4\, p\times r & (r) & \varepsilon_4\, q\times r & (i) \\
I: & -\varepsilon_1 .. \varepsilon_4\, r\times p & & -\varepsilon_1 .. \varepsilon_4\, r\times q & & \varepsilon_1 .. \varepsilon_4\, p\times p & & \varepsilon_1 .. \varepsilon_4\, q\times q & \\
R: & \pm\, r\times p & (r) & \pm\, r\times q & (i) & \pm\, p\times p & (r) & \pm\, q\times q & (r)
\end{array}
\end{array}\right\} \quad (3.16)$$

In this table (r) means real and (i) means imaginary, leaving aside the influence of the ε's. As we see in every pentad three real and two purely imaginary quantities occur. This is in accordance with a theorem due to EDDINGTON, stating that if a pentad with signature $+ + + + +$ is represented by matrices, always three and not more of them can be chosen real and the remaining ones purely imaginary [17].

From (3.16) we see that for the signatures $+ + + +$ and $- - - -$ no real or purely imaginary representations are possible. For the interesting signature $+ + + -$ we have two essentially different real representations (applying the permutation $2, 4 \rightarrow 4, 2$ in II and $3, 4, 1 \rightarrow 1, 3, 4$ in III')

$$\left.\begin{array}{lcc}
 & \text{II}^m & \text{III}^m \\
i_1 & r\times p & r\times p \\
i_2 & p\times 1 & p\times 1 \\
i_3 & r\times r & r\times r \\
i_4 & r\times iq & iq\times 1 \\
I & iq\times 1 & -r\times iq \\
R & \pm q\times 1 & \pm r\times q
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (3.17)$$

For the signature $- - - +$ we have to add a factor $\pm i$ in all products

[17]) This theorem of Eddington is a consequence of a much more general theorem that will be proved in the sequel.

of (3.17) and get then two purely imaginary representations. We call real and imaginary representations after MAJORANA[18]), who used the real ones for the description of neutral particles. The representation II^m was used by BELINFANTE[19]) who proved that it can be derived from the KRAMERS representation by a transformation $S \ldots S^{-1}$; $S = 1 \times 1 + iq \times q$ followed by $\underset{2}{i} \rightarrow -\underset{2}{i}$; $\underset{4}{i} \rightarrow i\,\underset{4}{i}$. II′, III, IV and IV′ lead to MAJORANA representations for the signature $++--$.

Writing out the representation I, and substituting 11, 22, 21 and 12 by 1, 2, 3 and 4 we get for $\lambda_1 = 1$, $\lambda_2 = 1$

$$\left.\begin{array}{lll}
\underset{1}{i}: & \underset{1}{\varepsilon} & ({}^{1}_{\;3} + {}^{4}_{\;2} + {}^{3}_{\;1} + {}^{2}_{\;4}) \\
\underset{2}{i}: & \underset{2}{\varepsilon} & ({}^{1}_{\;4} + {}^{4}_{\;1} - {}^{3}_{\;2} - {}^{2}_{\;3}) \\
\underset{3}{i}: & \underset{3}{\varepsilon}\, i & ({}^{1}_{\;3} + {}^{4}_{\;2} - {}^{3}_{\;1} - {}^{2}_{\;4}) \\
\underset{4}{i}: & \underset{4}{\varepsilon}\, i & ({}^{1}_{\;4} - {}^{4}_{\;1} - {}^{3}_{\;2} + {}^{2}_{\;3}) \\
I: & \underset{1}{\varepsilon}\ldots\underset{4}{\varepsilon} & ({}^{1}_{\;1} + {}^{2}_{\;2} - {}^{3}_{\;3} - {}^{4}_{\;4}) \\
R: & \pm & ({}^{1}_{\;1} + {}^{2}_{\;2} - {}^{3}_{\;3} - {}^{4}_{\;4})
\end{array}\right\} \quad \ldots\ldots \quad (3.18)$$

From this we get for the signature $++++$

$$\left.\begin{array}{l}
{}^{1}\!/_{2}(\underset{1}{i} + i\,\underset{3}{i}) : {}^{3}_{\;1} + {}^{2}_{\;4} \\
{}^{1}\!/_{2}(\underset{1}{i} - i\,\underset{3}{i}) : {}^{1}_{\;3} + {}^{4}_{\;2} \\
{}^{1}\!/_{2}(\underset{2}{i} + i\,\underset{4}{i}) : {}^{4}_{\;1} - {}^{2}_{\;3} \\
{}^{1}\!/_{2}(\underset{2}{i} - i\,\underset{4}{i}) : {}^{1}_{\;4} - {}^{3}_{\;2}
\end{array}\right\} \quad \ldots\ldots\ldots \quad (3.19)$$

If 2 and 4 are interchanged (that means $12 \rightarrow 2$; $22 \rightarrow 4$) these matrices correspond to the matrices H_4, H_2, H_3 and H_1 used by CARTAN[20]) and I corresponds to his matrix H_0. Hence CARTAN also uses a representation of the form I.

The representation I is the only one that can immediately be generalized for an R_n; $n = 2\nu$, as was shown by BRAUER and WEYL[21])

$$\left.\begin{array}{lll}
\underset{1}{i}: \underset{1}{\varepsilon}\, p \times 1 \times \ldots \times 1 & \underset{\nu+1}{i} \quad \underset{\nu+1}{\varepsilon}\, q \times 1 \times \ldots \times 1 & \\
\underset{2}{i}: \underset{2}{\varepsilon}\, r \times p \times 1 \ldots \times 1 & \underset{\nu+2}{i} \quad \underset{\nu+2}{\varepsilon}\, r \times q \times \ldots \times 1 & \\
\vdots \quad \vdots & \vdots \qquad \vdots & \text{(always} \\
\underset{\nu}{i}: \underset{\nu}{\varepsilon}\, r \times r \times \ldots r \times p & \underset{n}{i} \quad \underset{n}{\varepsilon}\, r \times r \times \ldots r \times q & \nu \text{ factors)} \\
I: (-1)^{\binom{\nu+1}{2}}\, i^{\nu}\, \underset{1}{\varepsilon} \ldots \underset{n}{\varepsilon}\, r \times r \times \ldots \times r & & \\
R: \pm r \times r \times \ldots \times r & &
\end{array}\right\} \quad (3.20)$$

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18) E. MAJORANA, Nuov. Cim. **14**, 171 (1937)

19) L.c. p. 10.

20) L.c. II p. 6.

21) L.c. p. 429.

These authors use $\lambda = 1$ in all M_2's. Contrary to the case $\nu = 1$ (3, 20) is not the most general representation satisfying the first invariant condition, because not every transformation $S \ldots S^{-1}$ in E_N leaving invariant the invariant E_M's of R can be obtained by changing the λ's in the factor-M_2's. But it is the only one that satisfies the additional condition that the r's in a wellchosen set of factor-M_2's have all the special form ${}^1{}_1 - {}^2{}_2$. Also for general values of ν CARTAN uses a representation of the form I.

(To be continued.)

Geology. — *Sur un massif granodioritique et ses phénomènes de contact à l'ouest de Palopo (Célèbes).* Par H. A. BROUWER.

(Communicated at the meeting of May 28, 1949.)

La route de Palopo à Rante Pao, dans l'extrême sud de la Célèbes centrale, traverse un massif granodioritique. ABENDANON [1]) l'a indiqué schématiquement sur sa carte géologique aux entours du B. Poeang, dont le sommet atteint une hauteur de 2023 mètres. Pendant mon retour en auto de Palopo à Macassar, à la fin de mes explorations à travers la Célèbes centrale [2]), j'ai recueilli le long de cette route des échantillons de roches du massif, du cortège filonien et de l'auréole de métamorphisme.

Les premiers affleurements du contact des granodiorites avec les roches encaissantes se trouvent à une distance d'à peu près 17 kilomètres de Palopo (fig. 1) [3]). Le contact occidental se trouve à une distance d'à peu près 28 kilomètres de Rante Pao. Les échantillons recueillis ne sont pas nombreux et ne permettent pas de faire une étude approfondie de l'auréole de métamorphisme et des variations de bordure du massif. Cependant, la région étant peu connue, certains résultats méritent d'être signalés.

La série sédimentaire (formation de Maroro) dans laquelle les granodiorites sont intrusives, se compose principalement de schistes argileuses, souvent rougeâtres, avec des intercalations de grès, de marnes et de calcaires. L'âge de la série a donné lieu à discussion; en partie elle est d'âge éocène et certains auteurs ont admis qu'elle représente une série compréhensive avec passage continu du crétacé au tertiaire.

Roches du massif.

Les échantillons étudiés sont pour la plupart des granodiorites et des diorites quartzifères assez finement grenues. Ils montrent souvent une texture gneissique plus ou moins accentuée. De grands cristaux de feldspath prêtent à certaines roches une apparence porphyroïde. Les échantillons recueillis loin du contact se rapprochent le plus du granite. Il sont riches en orthose et biotite. Dans les roches porphyroïdes de grands cristaux d'orthose englobent les autres éléments; l'orthose des roches non porphyroïdes forme des plages xénomorphes. Certaines roches plus près du contact sont plus riches en amphibole et plagioclase, mais les échantillons recueillis

[1]) E. C. ABENDANON. Voyages géologiques et géographiques à travers la Célèbes centrale (1909—1910). Avec atlas. Vol. I. Chap. I et IV.

Du temps des voyages d'ABENDANON à travers la Célèbes centrale la route n'était pas encore construite. Il a traversé la région plus au sud et plus au nord. Les roches intrusives, filoniennes et métamorphiques de sa collection, en partie des galets et des blocs roulés, ont été étudiées par GISOLF (voir E. C. ABENDANON, loc. cit. Vol. III).

[2]) H. A. BROUWER. Geologische Onderzoekingen op het eiland Celebes. Verh. Geol. Mijnb. Gen. X, 1934, p. 39—218. — Geological Explorations in the island of Celebes under the leadership of H. A. BROUWER, Amsterdam, 1947.

[3]) C (fig. 1) indique la localité la plus orientale où nous avons observé le contact.

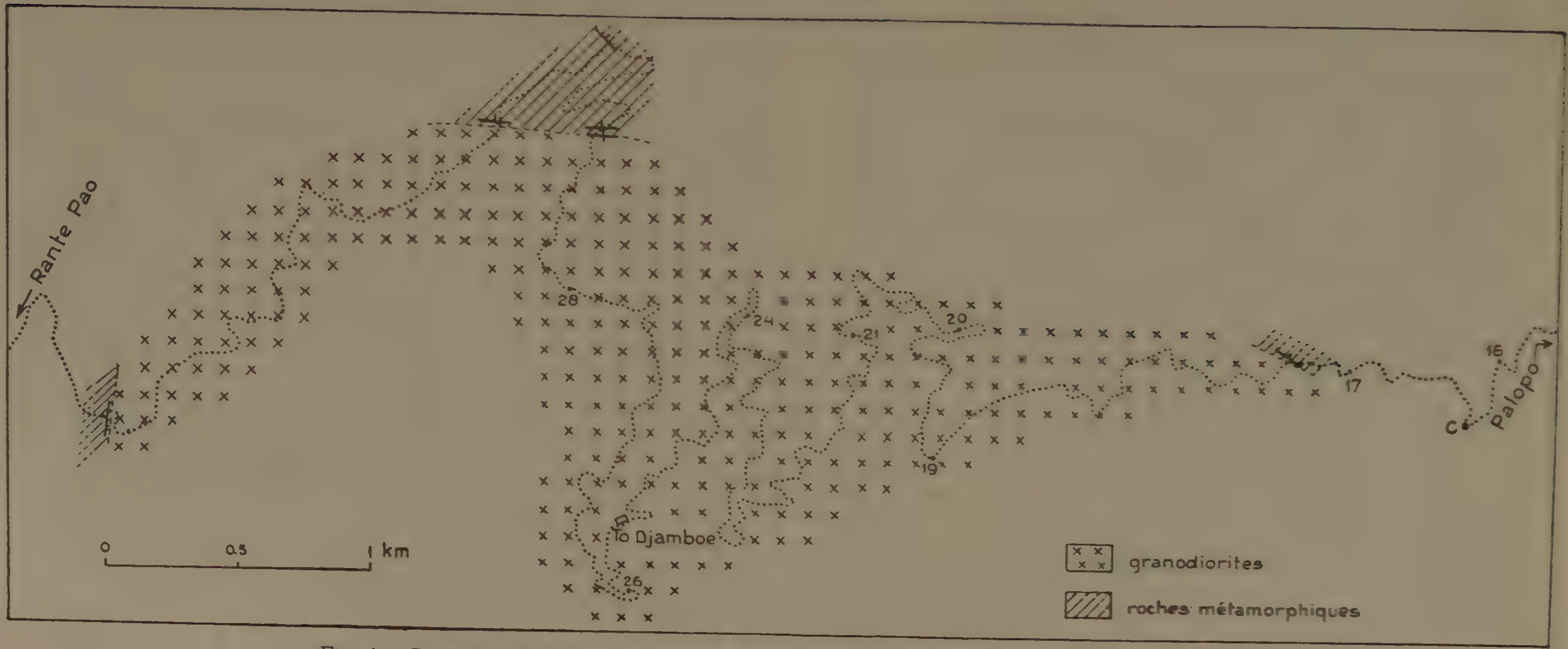

Fig. 1. *Granodiorites et roches métamorphiques le long de la route de Palopo à Rante Pao.*
16, 17 etc. distances en kilomètres de Palopo. La limite entre le massif et son auréole, de même que l'extension des granodiorites et des roches métamorphiques sont approximatives. Le bord du massif au contact oriental (C) n'est pas suffisamment connu et n'a pas été dessiné ici.

sont trop peu nombreux pour pouvoir juger à quel point le massif est plus dioritique vers sa bordure. Des enchevêtrements de biotite et amphibole, indiquant une cristallisation simultanée des deux minéraux, sont fréquents. Dans plusieurs roches nous avons observé l'orthite et le sphène.

Les échantillons recueillis au contact avec les roches encaissantes sont en majeure partie porphyriques avec des phénocristaux de feldspath, de quartz et de biotite, nageant dans une pâte finement grenue constituée par les mêmes éléments. Les phénocristaux de feldspath sont principalement des plagioclases. Outre ces minéraux il est d'autres éléments en petites quantités parmi lesquels la muscovite, le sphène et l'orthite. Près du contact occidental, dont nous n'avons pas étudié les détails, s'observent de petits affleurements isolés de roches porphyriques qui ne sont pas indiqués dans la figure 1. Ces roches sont caractérisées par une texture schisteuse accentuée. Au microscope elles ont l'aspect des gneiss oeillés sans texture cataclastique.

Nous admettons que les textures gneissiques, plus ou moins accentuées, sont toutes attribuables aux mouvements pendant la mise en place du magma encore plastique.

Roches de filon.

Près du contact septentrional nous avons recueillis des échantillons de filons, qui sont en rapport avec le massif granodioritique. Les filons recoupent les roches métamorphiques.

La roche filonienne le plus près du contact est de caractère lamprophyrique. Elle est riche en éléments ferromagnésiens, dont l'amphibole brune ou verdâtre prédomine. Les phénocristaux d'amphibole et de biotite nagent dans une pâte, qui contient les mêmes minéraux, associés à des plagioclases assez basiques, de l'orthose et du quartz.

Plus loin de la bordure du massif les roches métamorphiques sont recoupées par une microdiorite quartzifère. La roche montre des phénocristaux de plagioclase zoné, d'amphibole verte, de biotite brune et de quartz. La pâte contient les mêmes éléments auxquels est associé l'orthose.

Produits du métamorphisme.

Les échantillons étudiés ne montrent qu'un faible apport de substances émanées de la roche intrusive. La recristallisation des roches métamorphiques tient la place prépondérante et les produits du métamorphisme dépendent de la nature des sédiments métamorphisés.

Les roches recueillies au contact même sont des cornéennes, qui se sont produites aux dépens de roches silico-alumineuses et de roches marneuses. Dans les cornéennes de la première groupe le développement abondant de l'andalusite est caractéristique. Il s'y joint la biotite, le quartz, la cordiérite et le feldspath. Dans certaines cornéennes de minces lits superposés montrent une teneur variée en orthose. Dans d'autres un plagioclase acide est parmi les constituents. Comme le zircon, la rutile et la magnétite, la tourmaline est un minéral accessoire qui se trouve généralement en

petite quantité dans toutes les roches. Un des échantillons recueillis entre les poteaux kilométriques 20 et 21 [4]) est caractérisé par le développement abondant de tourmaline, montrant un important apport local des substances émanées de la roche intrusive.

Quand la roche sédimentaire est plus marneuse il se produit au contact des cornéennes de grain fin dont la composition est diverse. On les trouve au contact entre les poteaux kilométriques 28 et 29. Parmi les éléments blancs le quartz est prédominant. Augite, amphibole et biotite sont les minéraux caractéristiques de zones alternantes, dans lesquelles un ou deux de ces minéraux prédominent. Un autre échantillon montre des zones, essentiellement formées de diopside, quartz et feldspath, alternant avec des zones riche en épidote et pyrite. Une lentille dans la roche contient les minéraux suivants: sphène, pyrite, quartz, chlorite, actinolite et magnétite, accompagnant l'épidote prédominante.

Plus loin des contacts nous avons recueilli des schistes micacés et des schistes tachetés, comme dans le cas de beaucoup d'autres auréoles de métamorphisme. Nous signalons deux types qui se distinguent des types normaux. Le premier est un schiste à chiastolite riche en graphite; quartz, muscovite, tourmaline, leucoxène et rutile sont parmi les constituents de la roche. Le deuxième est un schiste à actinolite et biotite, qui contient du quartz, un plagioclase assez basique en partie séricitisé, du sphéne et de la magnétite.

Dans la partie traversée par la route de Palopo à Rante Pao il semble que la bordure du massif granodioritique se conforme à la direction des couches encaissantes. Près du bord du massif nous avons observé une zone à deux temps de consolidation. Les textures gneissiques, parfois très accentuées près du contact, sont attribuables aux pressions, exercées sur le magma pendant la phase plastique et à l'écoulement gêné près du contact.

Quant à l'auréole de métamorphisme la roche intrusive n'a pas agi d'une façon purement physique mais en général l'apport de substances émanées du magma est faible. Un des échantillons seulement contient de la tourmaline en abondance. Les feldspaths des cornéennes du contact indiquent tout au plus un faible métamorphisme par imbibition.

La texture porphyrique observée dans la zone de bordure du massif et le faible apport de substances dans les roches métamorphiques indiquent que les roches exposées appartiennent à un niveau élevé des montées granodioritiques. Plus au nord dans la partie occidentale de la Célèbes centrale on trouve des niveaux plus profonds où le métamorphisme plus intense est souvent accompagné d'injection [5]).

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4) Le long de cette partie de la route nous n'avons qu'observé des blocs roulés de roches métamorphiques, qui ne sont pas indiquées dans la figure 1. Le contact est probablement à proximité.

5) C. G. EGELER. Contribution to the petrology of the metamorphic rocks of western Celebes. Geol. expl. in the island of Celebes under the leadership of H. A. BROUWER, p. 334, Amsterdam 1947.

Zoology. — *Direct effects of isotonic and hypotonic lithium chloride solutions on unsegmented eggs of Limnaea stagnalis.* I. By CHR. P. RAVEN and J. R. ROBORGH. (Zoological Laboratory, University of Utrecht.)

(Communicated at the meeting of May 28, 1949.)

When eggs of *Limnaea stagnalis* are treated, in their capsules, with weak (0.001 %—0.01 %) solutions of LiCl, either exogastrulation occurs or embryos with synophthalmic and cyclopean head malformations may develop in a number of cases (RAVEN 1942). In these cyclopean monsters dorsomedian parts of the head, which in normal development arise from cells surrounding the original animal pole of the egg, are suppressed (RAVEN 1947). It was concluded from these observations that the influence of lithium may be explained by a depressing action on a gradient field with high point at the animal pole.

The action of lithium chloride is not altogether specific: both exogastrulae and cyclopean malformations may also be obtained with the chlorides of sodium, potassium, magnesium and calcium. However, in a quantitative respect the action of lithium greatly surpasses that of all other kations together (RAVEN and SIMONS 1948).

The action of lithium is phase-specific. A treatment immediately after oviposition leads to the formation of cyclopean malformations; at the time of 2nd cleavage there is a maximum susceptibility for the production of exogastrulae; whereas a second maximum for the production of head malformations exists at the 24-cell stage (RAVEN, KLOEK, KUIPER and DE JONG 1947; RAVEN and RIJVEN 1948).

Lithium chloride in hypertonic and slightly hypotonic concentrations decreases the viscosity and increases the tension at the surface of *Limnaea* eggs in the egg capsules; very weak solutions (about 0.01 %), on the contrary, produce a slight increase of viscosity (DE VRIES 1947).

In order to study the direct effects of lithium chloride on the *Limnaea* egg, in further experiments the eggs were exposed to the solutions after decapsulation.

M. GRASVELD (1949) showed that LiCl is more toxic than NaCl, KCl, $MgCl_2$ or $CaCl_2$. In no concentration of LiCl development proceeds beyond the 4—8 cell stage. Hypertonic LiCl solutions cause a great increase of amoeboid activity of the eggs.

DE GROOT (1948) studied the influence of LiCl solutions varying from 4 % to 0.05 %. Development does not proceed beyond the second cleavage. It may be inhibited at different stages, dependent on concentration, stage of treatment, temperature and susceptibility of the eggs. In 0.6—0.4 %,

the eggs show a very intense amoeboid activity during and after the formation of the second polar body in the controls.

The cytological consequences of the treatment were studied by DE GROOT, using concentrations of 1 %—0.2 % (the latter is about isotonic to the eggs). In 1 % solutions, the first maturation spindle sinks into the interior of the egg and degenerates. In 0.5 %, the first polar body is formed in a normal way, but then the second maturation spindle sinks into the interior and degenerates. In 0.4 % LiCl, development may be deflected at different moments. If treatment starts at an early stage, the chromosomes may swell into karyomeres shortly after the extrusion of the first polar body. No second maturation spindle is formed, but the sperm nucleus swells into a male pronucleus and rises to the animal pole which it may have reached already 35 minutes after the formation of the first polar body. The egg karyomeres coalesce to a female pronucleus which copulates with the male pronucleus; cleavage spindles appear in which the chromosomes show an abnormal arrangement.

In other batches, treated with 0.4 % LiCl, after the extrusion of the first polar body a second maturation spindle is formed, which sinks into the egg and places itself perpendicular to the egg axis. In still other batches, both polar bodies are formed in a normal way, but either the egg karyomeres are displaced towards the centre of the egg or development stops after copulation of the pronuclei.

The cytoplasmic differentiations are greatly disturbed by treatment with lithium chloride. The distribution of the vegetative pole plasm is very abnormal, especially in 0.2 % LiCl; the animal pole plasm is not formed in any of the concentrations studied.

When 24-cell stages of *Limnaea* are treated with lithium chloride (0.05—0.4 %), a swelling of the nuclei and a decrease of their distance from the cell surface occurs. Both phenomena are most pronounced in 0.05 % LiCl, and diminish in intensity with increasing concentrations of LiCl; in distilled water they are less pronounced, too (RAVEN and DUDOK DE WIT 1949).

Summarizing, it may be said that a treatment with lithium chloride affects both the nuclei and the cytoplasm of *Limnaea* eggs. However, since most of the solutions employed by DE GROOT are hypertonic to the eggs, it is not easy to decide which of the results obtained by him are due to hypertonicity and which are specific lithium effects. Therefore, in the present investigation the effects of isotonic and hypotonic lithium chloride solutions on decapsulated undivided *Limnaea* eggs have been studied.

Material and methods.

The snails were stimulated to oviposit by means of *Hydrocharis* in the usual way (RAVEN and BRETSCHNEIDER 1942). Immediately after oviposition, the eggs were decapsulated and washed three times in distilled water. Part of the eggs were kept in this medium as controls, the other

ones were transferred immediately to the LiCl-solutions. Solutions of 0.2 %, 0.15 %, 0.1 % and 0.05 % LiCl have been used. The former is about isotonic to the eggs, the latter three are hypotonic. After 1, 2, 3 and 4 hours part of the eggs, both of experimental and control series, were fixed in BOUIN's fluid. They were embedded in paraffin, sectioned at 5 μ, and the sections stained either with iron haematoxylin and saffranin or with azan.

In total, 495 treated eggs have been studied cytologically. Table I gives their distribution according to concentration and duration of treatment. They have been compared with 341 control eggs developed in distilled water.

TABLE I.

Concentration LiCl / Duration of treatment	0.20 %	0.15 %	0.10 %	0.05 %	Total
1 h.	47	20	26	23	116
2 h.	49	55	43	30	177
3 h.	3	35	38	29	105
4 h.	20	29	25	23	97
Total	119	139	132	105	495

In our previous investigations, azan-stained sections of BOUIN-fixed material proved to be very valuable for the study of cytoplasmic differentiations in the egg; especially the vegetative pole plasm showed a very clear elective staining in this way. However, in this year's preparations the azan staining did not give any satisfactory results, though various modifications of the technique have been tried. Therefore, the cytoplasmic differentiations have not been considered in this paper; this point needs further study with improved staining techniques.

Results.

The following effects of the treatment with isotonic and hypotonic LiCl solutions have been observed:

1. a swelling of the telophase chromosomes of the first maturation division into karyomeres, accompanied with an accelerated migration of the sperm nucleus towards the animal pole;
2. an increased hydration of the spermaster after the second maturation division, with a delay in its disappearance;
3. a considerable increase of amoeboid mobility of the eggs after the second maturation division;
4. a delay of the beginning of cleavage;
5. various abnormalities of the first cleavage mitosis.

1. *The swelling of chromosomes between the first and second maturation division.*

We will begin with a short description of the period between both maturation divisions in normal eggs (RAVEN 1945, 1949). The first maturation spindle possesses a well-developed aster at both ends. At the end of anaphase, the inner aster is big and has a large clear "central body". When the central group of dyads reaches the margin of the "central body", their movement stops; the 18 dyads arrange themselves along the outer surface of the "central body" into a more or less irregular ring. Simultaneously, the "central body" increases in size and transforms into the second maturation spindle. The dyads, which have retained their individuality and their compact structure, now arrange themselves into the equatorial plate of this spindle. At the outer end of the spindle, astral radiations are formed in the cytoplasm. The inner aster of the second maturation spindle, however, is provided by the spermaster. This has appeared during the telophase stage of the first maturation division, has grown rapidly in size during the formation of the second maturation spindle, and now fuses with the deep end of the latter. Then the second maturation division begins. Immediately after the second polar body has been extruded, the telophase chromosomes remaining in the egg begin to swell into karyomeres, which assemble immediately beneath the egg cortex at the animal pole. The remaining aster (the former spermaster) shifts to a deeper position; it has a big, clear, vacuolated "central body" and a ring of short astral rays; soon, however, it becomes inconspicuous and vanishes altogether. The sperm nucleus, which had retained a subcortical position and compact structure till the end of the second maturation division, begins to move towards the animal pole at the moment the egg chromosomes begin to swell into karyomeres. During its migration, it swells and develops into a male pronucleus, which meets the female karyomeres at the animal pole.

In the controls of the present investigation, which have developed in distilled water, these processes take place in an entirely normal way. No deviations in the course of the maturation divisions as compared with that in normal eggs can be observed.

Two points in this cycle of events must be especially stressed: first, the fact that the chromosomes (dyads) between both maturations divisions remain compact and are arranged as such into the second maturation spindle; secondly, that the sperm nucleus does not begin its migration and its transformation into the male pronucleus before the end of the second maturation division. In both respects, the eggs treated with lithium solutions differ from the controls.

In these eggs, the first maturation division takes place in a normal way (fig. 1 *a—b*). However, as soon as the dyads have reached the "central body" of the inner aster of the spindle and even before the first polar

body has entirely been pinched off, the chromosomes begin to swell. At first, they appear each surrounded by a clear space in the sections (fig. 1 *c*). Shortly after, they have formed small vesicles with a distinct wall, while the chromatin begins to disperse in the interior of the vesicle (fig. 1 *d*).

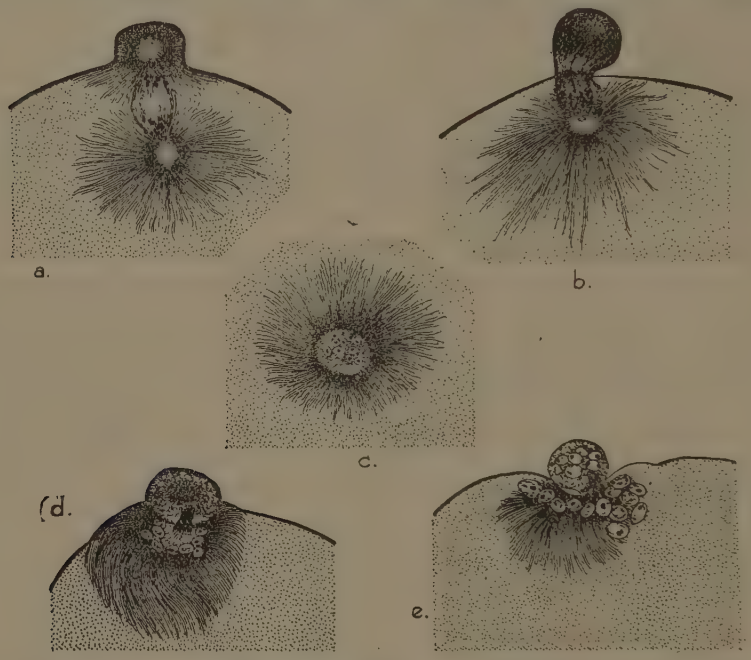

Fig. 1. *Limnaea stagnalis*, 0.05 % LiCl, 1 hour. *a*. Late anaphase, *b*. Telophase of first maturation division. *c*. Swelling of chromosomes. *d*, *e*. Karyomeres surrounding stalk of first polar body.

These karyomeres, which swell still further until the largest ones have reached a size of about $7\frac{1}{2} \times 6\frac{1}{2}\ \mu$ (fig. 1 *e*), surround the connecting stalk of the first polar body, which has been condensed into a heavily-staining "mid-body", forming an irregular ring around its base. They are surrounded, in their turn, by the astral radiations of the inner maturation aster, which can be followed quite a distance into the cytoplasm. The centre of this aster, between the ring of karyomeres, is occupied by an area of rather dense protoplasm, representing the "central body" of the aster; this is hardly visible, however, because it is flattened against the surface, contrary to its position in normal eggs of this stage, where it is situated at some distance beneath the surface and forms a round or ellipsoid body.

The chromosomes of the first polar body, like those of the egg, swell into karyomeres, be it somewhat delayed as compared with the egg chromosomes (fig. 1 *e*).

This abnormal development is not restricted to a certain concentration of LiCl only; we have observed it both in isotonic (0.20 %) and in hypotonic (0.10 %, 0.05 %) solutions. Moreover, DE GROOT (1948), as

stated above, found a similar phenomenon in eggs treated with a hypertonic (0.40 %) LiCl solution. On the contrary, in the control eggs in distilled water no swelling of chromosomes occurs at this stage. Hence, this swelling is not a purely osmotic phenomenon, but must be due to a specific action of the LiCl.

The swelling of the maturation chromosomes into karyomeres after the extrusion of the first polar body resembles the process normally occurring after the second maturation division. It is, therefore, interesting to observe that it is accompanied with another phenomenon taking place in normal development simultaneously with the formation of egg karyomeres: the migration of the sperm nucleus towards the animal pole and its transformation into a male pronucleus. Though the sperm nucleus is not always easily detectable in the eggs, in all cases in which it is found it is situated at this stage as a compact dark body beneath the egg cortex in the controls. On the contrary, in those Li-treated eggs in which the first maturation chromosomes have formed karyomeres, the sperm nucleus without exception has begun its migration towards the animal pole, which it has reached already in a number of cases, lying immediately beside or even in the peripheral part of the maturation aster surrounding the karyomeres. At the same time, it has swollen into a big vesicular pronucleus, in which the individual chromosomes have, evidently, retained some individuality so that it looks as if it were composed of a number of fused karyomeres.

This accelerated swelling and migration of the sperm nucleus, which has also been observed already by DE GROOT in similar eggs, appears to be strictly correlated with the swelling of the egg chromosomes into karyomeres. The fact that both phenomena show the same synchronicity in normal development, be it at a later stage, strongly speaks in favour of the supposition that they are causally related. Presumably, the egg chromosomes and the sperm nucleus react similarly to a certain condition of the cytoplasm by swelling, and the sperm nucleus only begins to respond to the attractive forces directing it towards the egg karyomeres or towards the animal pole after it has undergone some swelling. The effect of the Li-treatment could be explained, in this case, by the supposition that it induces in the cytoplasm the condition provoking the swelling of the karyomeres and sperm nucleus.

In DE GROOT's eggs in 0.4 % LiCl, in which karyomeres were formed after the extrusion of the first polar body no second maturation spindle developed. The karyomeres coalesced to a female pronucleus which copulated with the male pronucleus. A cleavage spindle with abnormal arrangement of the chromosomes formed afterwards.

One might expect a similar development to occur in our eggs in 0.2—0.05 % LiCl with premature karyomere formation. However, it appears that this is not the case. Of course, there is no direct evidence as to what might have become of eggs like those of fig 1 *d* and *e*, had they be allowed to develop further. However, the following indirect evidence is available:

1°. according to DE GROOT (1948) and M. GRASVELD (1949), in isotonic and hypotonic solutions of LiCl both polar bodies are formed;

2°. in no batches fixed 2—4 hours after the beginning of treatment we ever observed any indication that the second maturation division had been suppressed.

It seems, therefore, that the premature swelling of the chromosomes into karyomeres is a reversible process, which may be followed by a normal second maturation division. Some of our batches seem to show in which way this takes place. In batch *C 4—1* (0.10 % LiCl, 1 hour), some eggs still show a cluster of karyomeres surrounding the stalk of the first polar body in the centre of a large maturation aster (fig. 2*a*). In other eggs, the astral radiations have been reduced, the "central body" is situated somewhat deeper and has rounded off; the chromosomes are arranged in a circle around it (fig. 2*b*). Though they still have a swollen

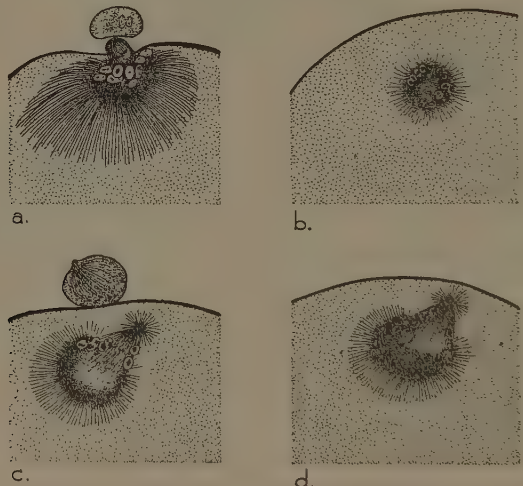

Fig. 2. *Limnaea stagnalis,* 0.10 % LiCl, 1 hour. *a.* Karyomeres of first maturation division. *b.* Swollen chromosomes surrounding central body. *c, d.* Formation of second maturation spindle. Chromosomes still somewhat swollen.

appearance, the chromatin in each of them has condensed again into a compact body. Fig. 2*c* and *d* show further transitional stages in the formation of the second maturation spindle ("acorn-stage" of the latter); the chromosomes still are somewhat swollen. In batch *P—1* (0.20 % LiCl, 1 hour), besides eggs with karyomeres (fig. 3*a*) also normal-looking anaphase and telophase stages of the second maturation division (fig. 3 *b—d*) have been found, in which the chromosomes have the appearance of compact dark bodies. Apparently, therefore, the swelling of the chromosomes immediately after the completion of the first maturation division is followed by a deswelling when the second maturation spindle forms.

Some points in this cycle of events are not yet entirely clear and make

a further study desirable. This is especially the case as regards the fate of the male pronucleus after its premature formation and migration. Though we have paid special attention to this point, we have not yet been able to find out what becomes of the male pronucleus when the

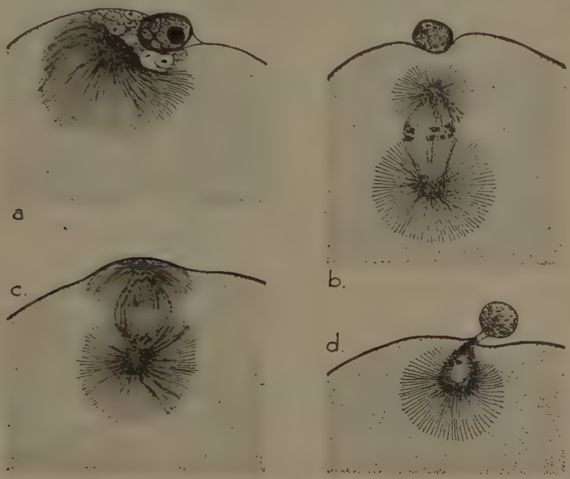

Fig. 3. *Limnaea stagnalis,* 0.20% LiCl, 1 hour. *a.* Karyomeres of first maturation division. *b.* Anaphase, *c.* Telophase of second maturation division. *d.* Extrusion of second polar body.

deswelling of the karyomeres occurs and they arrange themselves again into the maturation spindle. If it is true that egg chromosomes and sperm nucleus react in the same way to the condition of the cytoplasm, it might be supposed that the deswelling of the karyomeres will be accompanied with a similar process in the male pronucleus; in this case the latter, reverted to the state of a compact strongly basophil body, will be extremely difficult to detect among the basophil granulations of the egg cytoplasm.

Summarizing, it may be concluded that a first effect of the Li-treatment is a temporary change in the state of the cytoplasm, bringing about the swelling of both egg chromosomes and sperm nucleus, which in its turn causes their mutual attraction and the migration of the male pronucleus towards the animal pole. In isotonic and hypotonic solutions it is soon followed by a phase, in which again the deswelling influences prevail and the egg chromosomes return to their normal condition.

2. *Increased hydration of the spermaster.*

As has been stated above, in normal development the spermaster, after its temporary fusion with the deep end of the second maturation spindle, becomes independent again after the formation of the second polar body and shifts to a deeper position. Its "central body" meanwhile has greatly enlarged; it forms a big, clear, somewhat vacuolated space, surrounded by a ring of denser granular cytoplasm, in which a few short astral rays are visible for some time (fig. 4*a*). The latter soon become blurred, the

difference between the clear central area and the darker ring fades away and after some time only a somewhat clearer area in the cytoplasm indicates the position of the former spermaster; then this disappears too. The sperm nucleus, during its migration towards the animal pole, may temporarily be located within the spermaster or in its margin (fig. 4*a*).

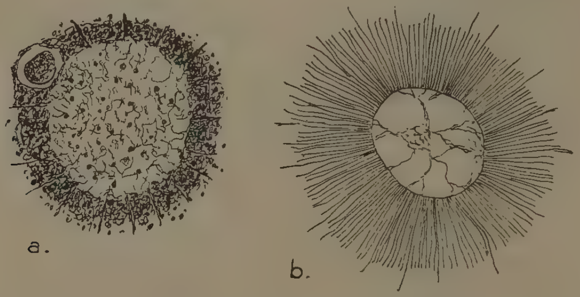

Fig. 4. Spermaster shortly after extrusion of second polar body *a*. Control egg. *b*. Egg treated with 0.20 % LiCl, 1 hour.

In the controls of the present experiments, the appearance and evolution of the spermaster is entirely the same as that just described. In the eggs, which had been treated with 0.20—0.15 % LiCl, however, the spermaster showed a different picture (the fixation and staining techniques being the same). The central area ("central body") is much more highly vacuolated in this case; the big vacuoles, which appear empty in the sections, are separated by fine protoplasmic meshes. Externally, this central area appears to be bounded by a definite fine protoplasmic lamella against the dense rim surrounding it. In eggs fixed shortly after the extrusion of the second polar body, this rim consists entirely of tightly packed astral rays, fine but very distinct and of moderate length (fig. 4*b*). Later, the astral radiations gradually fade away, but the highly vacuolated central area remains visible for a considerable time as a definite structure. As a matter of fact, the disappearance of the spermaster is greatly delayed, at least in part of the eggs, as the following batches show:

B 6—2 (0.15 % LiCl, 2 hours): controls 24 eggs, spermaster disappeared in all. 30 Li-eggs, 14 with distinct highly vacuolated spermaster, 16 spermaster disappeared.

B 4—3 (0.15 % LiCl, 3 hours): 23 controls, no spermaster left. 23 Li-eggs, 12 with highly vacuolated spermaster, 11 without spermaster.

We may conclude, therefore, that treatment with 0.20—0.15 % LiCl solutions leads to an increased hydration of the central area of the spermaster and a delay in its disappearance. This effect only becomes visible after the extrusion of the second polar body; as a matter of fact, in earlier stages neither the spermaster nor the other asters of the maturation spindles exhibit any difference as compared with those of the controls.

(To be continued.)

Physics. — *Influence of the texture of the original matrix on the number of inclusions in aluminium single crystals obtained by recrystallization.* By W. G. BURGERS and V. CH. DALITZ. (Laboratorium voor Physische Scheikunde der Technische Hogeschool, Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of May 28, 1949.)

1. Some years ago BURGERS and MAY (1945) observed that aluminium single crystals often contain a large number of small inclusions, which, judging from their identical reflection after etching, possess definite preferential orientations. These inclusions apparently cannot be removed, or in any case only with difficulty, even after prolonged heating at the highest possible temperature. So for example in crystals prepared by us, heating at 630° C during 1000 hours did not cause a noticeable decrease in number. CARPENTER and TAMURA (1927) and also SEUMEL (1936) observed a decrease only after very long periods of heating.

Both by etching and by X-rays it was found by TIEDEMA, MAY and BURGERS (1948; 1949) that in far the most cases the lattice orientation of the small included crystallites [called "cristaux insulaires" by LACOMBE and BERGHEZAN (1949)] was approximately, with deviations up to 5—10°, that of one of the four possible spinel twins with regard to the surrounding crystal. From the fact that the Laue-spots were slightly elongated, it could be deduced that the inclusions are grains of the original matrix, at the cost of which the large crystals have grown, and which were not consumed by these crystals during their growth. It seems obvious to ascribe this behaviour to the approximately symmetrical lattice position of growing and non-consumed crystal, which reduces the tendency of the boundary to displace itself in one or other direction. This conclusion seems to be supported by a recent paper by DUNN and LIONETTI (1949) on the effect of orientation difference on grain boundary energies, the result of which points to a low value for the relative surface energy between two lattices in either parallel [1]) or in mutual twin orientations [2]).

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1) In this connection it is important to mention that, as shown by X-rays, a crystal grown by recrystallization actually contains also inclusions which, within a few degrees, have the *same* orientation as that of the surrounding crystal [TIEDEMA, MAY and BURGERS (1949)]. Moreover it was found [TIEDEMA (1949)] that a large crystal cannot grow at the expense of a texture with approximately the same orientation as the growing crystal.

2) SMITH (1948) pointed out that the extremely small energy of the twin boundary in annealed face-centered cubic metals in relation to the grain boundary can be deduced from the fact that a twin can meet a grain boundary at virtually any angle without much deviation of the latter.

Up till now we donot yet know whether a deviation of the exact twin position in an "arbitrary direction" is sufficient to impede boundary displacement or whether a deviation in one direction is more active in this respect than in others. We are inclined to believe that the latter supposition is the more probable and that the non-consumability of various grains is different, depending on their precise orientation relationship with regard to the growing crystal. We think this conclusion may be deduced from the experimental fact, illustrated in fig. 1 [3]), that the number of inclusions at the boundary of a growing crystal is markedly greater than more inside the crystal. This suggests that of those grains which are left unabsorbed in the first moment of their contact with the growing crystal, a part is still consumed in the course of the annealing process.

2. If the conclusion put forward in the foregoing is correct, one may expect it to be a necessary condition for the occurrence of inclusions in large crystals grown by recrystallization that the crystal growing at the expense of the fine-grained matrix should meet in the course of its growth grains in approximate twin position. If the grains of the original matrix exhibit none or in any case only a slightly pronounced preferential orientation, this possibility is always present, independant of the orientation of the new growing crystals. On the other hand, if the large crystals grow in a matrix with a very "narrow" range of orientations of its constituent grains, it may be anticipated that a new crystal either has practically no inclusions or on the contrary a very large number. The former would occur if the orientation of the new crystal is very much different from that of a spinel twin orientation with regard to the average position of the texture; the latter, if its orientation lays close to the twin orientation of the matrix.

In order to test this expectation, a matrix with a very narrow texture was prepared. To this end, starting from quasi-isotropic fine-grained material, large crystals (several cm^2 in area) were prepared according to CARPENTER and ELAM's original method (CARPENTER and ELAM, 1921). Practically all of these crystals contained a considerable number of inclusions, on the average 30—100 per cm^2. Figures 2 and 3 (crystal A) give examples. The inclusions were resistent against prolonged heating at 630° C.

When such crystals (we shall call them "mother-crystals") are subjected to an elongation of 10—15 %, they give rise to very pronounced textures (that of the deformed single crystals), which, on annealing, are transformed anew into large crystals. For example an extended single crystal plate of 8×2.5 cm^2 (thickness 0.5 mm) was transformed, by recrystallization at 550° C during one hour, into 1—7 new crystals, the number in each case depending on the exact amount of stretching.

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[3]) Also in fig. 9 in the paper by BURGERS and MAY (1945).

W. G. BURGERS and V. CH. DALITZ: *Influence of the texture of the original matrix on the number of inclusions in aluminium single crystals obtained by recrystallization.*

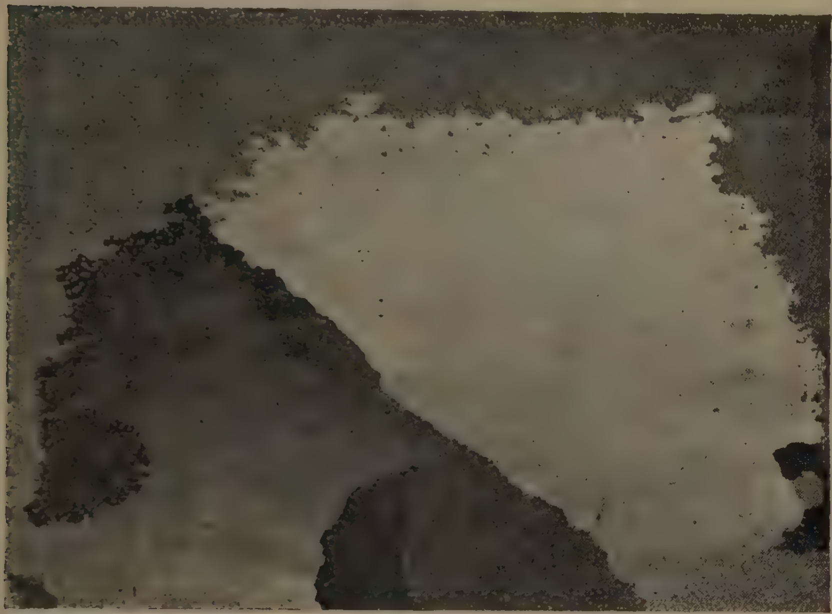

Fig. 1. Large aluminium crystals, growing into a fine-grained matrix. The large crystals show many inclusions ("insular grains"), which are non-absorbed grains of the original material. Their number is far greater close to the boundary of the growing crystal than in the centre of the crystal. The irregular shape of the boundary is very pronounced, due to the selective character of the growth process, as if the growing crystal were feeling its way in the fine-grained matrix. (½ Natural Size).

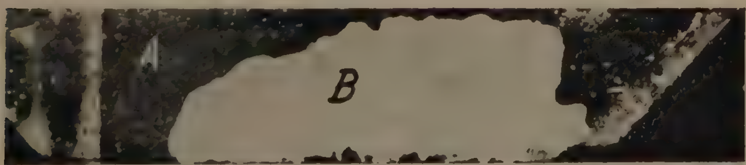

Fig. 2. New crystal (B) grown at the cost of a deformed single crystal (A). Contrary to (A), (B) is practically free from inclusions. (Natural Size).

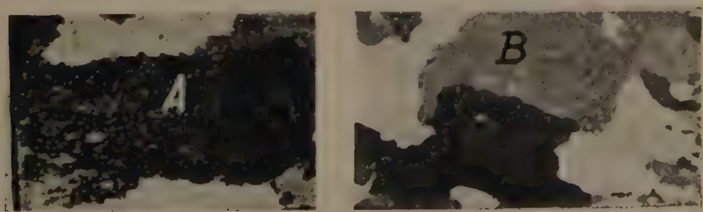

Fig. 3. The figure shows two parts of the same plate, which originally contained one large crystal (A), with many inclusions. The plate, after 20 % stretching, was cut into two parts, of which that shown at the right was recrystallized. One of the new crystals, (B), again contained a large number of inclusions. It was found that the orientation of (B) was approximately that of a spinel twin with regard to (A). (Natural Size).

Now it is obvious that in general neither the deformed texture of the "mother-crystal", nor the inclusions present in it, occupy a twin position with regard to the second generation of large crystals, so that we may expect these latter crystals to be free of inclusions. This proved actually to be the case for most crystals. Fig. 2 shows an example of such an "inclusion-free" crystal (B) grown into a mother-crystal (A) with many inclusions.

If, however, the particular case occurs that a crystal of the second generation happens to stand in (approximate) twin position with regard to the deformed mother-crystal, then the new crystal has a good chance to encounter, in the course of its growth, lattice regions in (approximate) twin position and the crystal may be expected to contain numerous inclusions. Actually such a case was found for the crystal shown in fig. 3. The two plates shown here formed originally one large plate, in which the "mother-crystal" (A) extended over the whole plate. After being cut into two halves, that shown at the right was stretched 20 % and recrystallized. The large new crystal (B) contains numerous inclusions. Laue-photographs of the (deformed) mother-crystal (A) and the new crystal (B) showed definitely that (B) occupied approximately a twin orientation with regard to (A).

From these experiments it may be concluded that actually the number of inclusions depends on the "width" of the texture of the original matrix in which the large crystals grow. In case this texture is sufficiently narrow, one obtains *either* (and presumably in most cases) crystals with none or only few inclusions *or* on the contrary (presumably rather seldom) crystals with very many inclusions. In case the original fine-grained matrix has no preferential orientation (is quasi-isotropic), the large crystals grown in it may be expected to have on the average the same number of inclusions (per unit of surface).

3. Finally the following remark may be made. As is well known, copper crystals, made by recrystallization of fine-grained material, in most cases contain numerous small twin crystals. It was found by TAKEYAMA (1930) that in some cases the number of twins could considerably be diminished by subjecting the primary crystals to a second deformation and recrystallizing anew. The same result was obtained by us in analogous experiments (although the elimination was far from complete). It seems justified to ascribe this behaviour to the same cause as valid in the case of aluminium.

Summary.

Large crystals of aluminium grown by recrystallization in fine-grained material often contain many "inclusions". These inclusions are unabsorbed grains of the original matrix, which are standing approximately in twin position with regard to the surrounding crystal. Their presence is due to the fact that a growing crystal cannot, or, if so, with great difficulty absorb

lattice regions in approximate twin position. It follows that the number of inclusions depends on the texture of the fine-grained material in which the crystals grow. If this texture is very narrow, crystals can grow which are practically free from inclusions.

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Physics. — *Straight twin lamellae in aluminium single crystals.* By V. CH. DALITZ and W. G. BURGERS. (Laboratorium voor Physische Scheikunde der Technische Hogeschool, Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of May 28, 1949.)

The occurrence of twin lamellae with straight boundaries, so common in recrystallized copper, nickeliron, α-brass and other metals, has been observed for aluminium only in relatively few cases. A striking example was found by ELAM (1928) in a specimen obtained by recrystallizing a stretched (10 %) single crystal. Recently LACOMBE and BERGHEZAN (1949) observed the occurrence of a few small lamellae with straight boundaries.

In the course of recrystallization experiments with aluminium single crystals deformed by extension, we happened to obtain after annealing the structure shown at about *natural size* in the accompanying photograph. Inside the large crystal 1, a set of practically parallel straight lamellae (2), extending over several centimeters, can be seen. X-rays (Laue-photographs) show that the dark and light parts stand in exact spinel twin relationship, the straight lines being (as also in the other cases mentioned) the traces of the twin plane (111) with the surface of the plate.

It is not yet known what causes the occasional formation of this banded structure type in aluminium. Twins in aluminium with *curved* boundaries may occur frequently in recrystallized polycrystalline testpieces, due to a process which was called "stimulation" by the growing crystal of lattice parts in the original matrix, which happened to possess a twin orientation with regard to the growing crystal [SANDEE (1942); BURGERS (1942)].

Several observations support the supposition that the occurrence of straight twin bands is related to the presence of a pronounced preferential orientation in the recrystallizing material, so for example the experiments performed by COOK and RICHARDS (1940) and COOK and MACQUARIE (1938) with copperstrips possessing [100] (100) orientation.

The fact that recrystallized polycrystalline aluminium seldom contains such a pronounced texture might account for the rare appearance of straight bands with this metal. A pronounced texture can, however, be obtained by deforming single crystals, as in the example shown, which (like the one found by ELAM) is produced by recrystallizing such a matrix.

In our case the direction of stretching was parallel to a [112]-direction of the original crystal. Also in some other crystals stretched parallel to this direction, straight twin bands were observed after recrystallization, although extending over a smaller length (only a few mm). We cannot say at the moment whether or not this result has a more general meaning.

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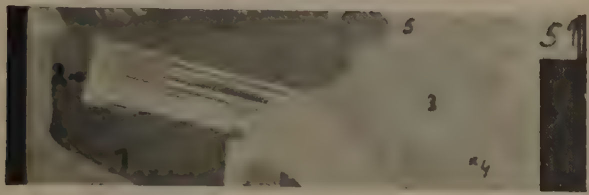
V. CH. DALITZ and W. G. BURGERS: *Straight twin lamellae in aluminium single crystals.*



a.

b.

Fig. 1. Aluminium plate with large crystals, obtained by recrystallization of a stretched (12.5 %) single crystal.

Crystal 1 shows straight bands (2) with sides parallel to a (111)-plane. The white and dark regions have mutually orientations corresponding to that of a spinel twin with regard to that plane.

a. front view
b. back view } (Natural size).

Mathematics. — *On Plimpton 322. Pythagorean numbers in Babylonian mathematics.* By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of May 28, 1949.)

NEUGEBAUER and SACHS gave an interpretation of the tablet Plimpton 322 (M.C.T. 1945) from the point of view, that the numbers on this tablet were obtained as a series solutions of the Pythagorean equation

$$d^2 = l^2 + b^2$$

in integers under the extra condition, that the proportion d/l changes from step to step by an, apparently, constant number. Their final result was, that the table was constructed by selecting numbers p/q and q/p from multiplicationtables so that

$$d/l = \tfrac{1}{2}\,(p/q + q/p)$$

satisfies the extra condition.

In our opinion a much simpler interpretation is possible, in which the production of Pythagorean numbers by using only one parameter — a method which seemed to be rejected according to NEUGEBAUER and SACHS — is the starting point.

The extra condition proves then to be absent, the "constant decrease" being merely accidental and the difficulties in terminology disappear.

Analysis.

1. If we put $l = 1$ the Pythagorean equation becomes

$$1 = d^2 - b^2 = (d + b)\ (d - b)$$

So when $d + b = \lambda$ we have $d - b = 1/\lambda$ and $d = \tfrac{1}{2}(\lambda + 1/\lambda)$, $b = \tfrac{1}{2}(\lambda - 1/\lambda)$, $l = 1$ are rational Pythagorean numbers, whenever λ is a rational number.

If therefore we form a table of sum and difference of the columns of a reciprocal table we have in

$$d = \lambda + 1/\lambda \qquad b = \lambda - 1/\lambda \qquad l = 2$$

Pythagorean triples.

e.g. $2,24 \quad 0,25 \quad d = 2,49 \quad b = 1,59 \quad (l = 2)$.

2. If we leave apart the constant l the numbers $\lambda + 1/\lambda$ and $\lambda - 1/\lambda$

are in general not relatively prime. Whether a *regular* common divisor (factors 2, 3 or 5) is present or not is evident from the last sexagesimal.

e.g. $\lambda = 2, 22, 13, 20$	$0, 25, 18, 45$	
$d = 2, 47, 32, 5$	$b = 1, 56, 54, 35$	both divisible by 5 (of the last sexagesimal),
$d = 33, 30, 25$	$b = 23, 22, 55$	both divisible by 5,
$d = 6, 42, 5$	$b = 4, 40, 35$	both divisible by 5,
$d = 1, 20, 25$	$b = 56, 7$	final numbers, reduced values;
Or $\lambda = 2, 8$	$0, 28, 7, 30$	
$d = 2, 36, 7, 30$	$b = 1, 39, 52, 30$	both divisible by 30, thus multiply by 2,
$d = 5, 12, 15$	$b = 3, 19, 45$	both divisible by 15, or multiply by 4
$d = 20, 49$	$b = 13, 19$	final numbers. ib — si.

3. NEUGEBAUER published M.K.T. I—16—23, discussing AO 6456, a complete six-place table of reciprocals. Taking all four-place values from this we find in the first and second column of Table I the reciprocals, the

TABLE I.

λ	$1/\lambda$	No.	d	b	Reduction
2, 24	0, 25	1	2, 49	1, 59	1
2, 22, 13, 20	0, 25, 18, 45	2*	*1, 20, 25*	56, 7	: 125
2, 20, 37, 30	0, 25, 36	3	1, 50, 49	1, 16, 41	×2 : 3
2, 18, 53, 20	0, 25, 55, 12	4	5, 9, 1	3, 31, 49	: 32
2, 18, 14, 24	0, 26, 2, 30	*	—	—	—
2, 15	0, 26, 40	5	1, 37	1, 5	×3 : 5
2, 13, 20	0, 27	6	8, 1	5, 19	×3
2, 10, 12, 30	0, 27, 38, 52, 48	—	—	—	—
2, 9, 36	0, 27, 46, 40	7	59, 1	38, 11	×3 : 8
2, 8	0, 28, 7, 30	8	20, 49	13, 19	×2 : 15
2, 6, 33, 45	0, 28, 26, 40	—	—	—	—
2, 5	0, 28, 48	9	12, 49	8, 1	×5
2, 2, 52, 48	0, 29, 17, 48, 45	*	—	—	—
2, 1, 30	0, 29, 37, 46, 40	10	2, 16, 1	1, 22, 41	×9 : 10
2	0, 30	11	*5* [1,15]	*3* [0,45]	×2 [: 2]
1, 57, 11, 15	0, 30, 43, 12	—	—	—	—
1, 55, 12	0, 31, 15	12	48, 49	27, 59	: 3
1, 53, 46, 40	0, 31, 38, 26, 15	—	—	—	—
1, 52, 30	0, 32	13	4, 49	2, 41	×2
1, 51, 6, 40	0, 32, 24	14	53, 49	29, 31	×3 : 8
1, 48	0, 33, 20	15	53	*28*	×3 : 8

The * denotes the reciprocals missing in *AO* 6456.
The *cursive* numbers are corrections necessary. No. 2, No. 6 are the same as those of NEUGEBAUER and SACHS, No. 15 differs from that in M. C. T: we divide 56 by 2 instead of multiplying 53 by 2.
The last column gives a process of reduction, to obtain d, b from $\lambda + 1/\lambda$ and $\lambda - 1/\lambda$. With the exception of No. 2 (and 11) all pairs d, b are relatively prime.

reduced values of $\lambda + {}^1/_\lambda$, $\lambda - {}^1/_\lambda$ forming the fourth and fifth column and numerating the reciprocals according to the third column, we see the columns IV, III, II of Plimpton 322. Apart from No. 11, which is given as 1, 15, 0, 45, *the values for* $l = 1$, and which hardly need a reduction, the proportion 5 : 3 being evident, we see some regular numbers "too much". These numbers are however of very high order. Writing $2^\alpha\, 3^\beta\, 5^\gamma$ (α, β, γ) we have

TABLE II.

* 2, 18, 14, 24	from	(1, 1, 6)	or	(11, 5, 0)
2, 10, 12, 30	from	(1, 1, 7)	or	(13, 6, 0)
2, 6, 33, 45	from	(0, 6, 4)	or	(12, 0, 2)
* 2, 2, 52, 48	from	(0, 4, 7)	or	(14, 3, 0)
1, 57, 11, 15	from	(0, 3, 6)	or	(12, 3, 0)
1, 53, 46, 40	from	(0, 7, 5)	or	(14, 0, 2).

It must be remarked, that the numbers with * are also missing in AO 6456. On the other hand the number missing in AO 6456: 2, 22, 13, 20 has its corresponding value on Plimpton 322, accepting the interpretation given here. This last number is obtainable from (0, 6, 3). The number of factors of all the others does also not exceed 13 and never more than three factors 5 occur:

(0, 0, 2); (0, 6, 3); (9, 1, 0); (7, 6, 0); (0, 3, 1); (0, 3, 0); (5, 5, 0); (7, 0, 0); (0, 0, 3); (1, 6, 1); (1, 0, 0); (8, 3, 0); (5, 0, 0); (3, 5, 0); (2, 3, 0).

As Plimpton 322 is about a millenium older than AO 6456 it is not unlikely that the high degree reciprocals were not yet contained in the table which the writer of the Plimtontablet had at his disposal. *But his table was complete under the conditions* $\alpha + \beta + \gamma \leq 13$; $\gamma \leq 3$.

Moreover the first column now contains the square of the diagonal minus unity for the case $l = 1$. *We thus see, that it is not necessary to complete the Plimpton tablet in column I*, as NEUGEBAUER and SACHS do, *by a unit which appears nowhere on the tablet*. Interpreting the first column as

$$\{\tfrac{1}{2}(\lambda + {}^1/_\lambda)\}^2 - 1$$

has the consequence that *takiltu* can be used in its ordinary meaning of square and the < mathematical > translation of the inscription above column I has to be: the square of the diagonal from which a unit has been substracted [1]).

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[1]) None of the missing numbers, Table II, is a divisor of 60^5. In addition to all divisors of 60^5 in the interval the table contains the three place reciprocals of 500.000, 512.000, 640.000 i.e. $\gamma \leq 3$.

P. VAN DER MEER discussing the inscription above column I indicated to me that the damaged sign between *i* ... *ú* could hardly be other than the polyvalent having among its values *seḫ*. So it is possible to read sag *iseḫú*, the side which fixes, determines the problem, which has been put equal to unity. This would give the key-stone of the interpretation.

Conclusion.

1. We have to read the columns from right to left.
2. The first column contains the ordinal numbers of reciprocals in a complete four-place table, not containing (α, β, γ) for $\alpha + \beta + \gamma > 13$, $\gamma > 3$, from 2, 24 — as near as possible to $1 + \sqrt{2}$ — to 1, 48.
3. The "solvent numbers" in the second and third column are the reduced values of diagonal and side obtained from $\lambda + {}^1/_\lambda$, $\lambda - {}^1/_\lambda$, by canceling common factors 2, 3, 5.
4. The first column contains the square of the diagonal diminished by unity if one of the sides is put equal to unity.
5. The constant decrease pro step of about one sixtieth in d/l is merely accidental. *There is no extra condition. The tablet contains everything that can be obtained from the four-place table mentioned above.*

Mathematics. — *On the minimum determinant of a special point set.* By K. MAHLER (Manchester). (Communicated by Prof. J. G. VAN DER CORPUT.) *)

(Communicated at the meeting of April 23, 1949.)

In a preceding paper [1]) C. A. ROGERS proves the inequality

$$\lambda_1 \lambda_2 \ldots \lambda_n \Delta(K) \leqslant 2^{\frac{n-1}{2}} d(\Lambda)$$

for the successive minima $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an arbitrary point set K for a lattice Λ. In the present paper, I shall construct a point set for which this formula holds with the *equality sign*. I prove, moreover, that there exist *bounded star bodies* for which the quotient of the two sides of ROGERS's inequality approaches arbitrarily near to 1. *The constant* $2^{\frac{n-1}{2}}$ *of* ROGERS *is therefore best-possible, even in the very specialized case of a bounded star body.*

1) Let $\mathfrak{R}_n$ be the n-dimensional Euclidean space of all points

$$X = (x_1, x_2, \ldots, x_n)$$

with real coordinates. For $k = 1, 2, \ldots, n$, denote by Γ_k the set of all points

$$(g_1, g_2, \ldots, g_k, 0, \ldots, 0)$$

with integral coordinates satisfying [2])

$$g_k \neq 0, \quad gcd(g_1, g_2, \ldots, g_k) = 1,$$

and by C_k the set of all points

$$X = tP, \text{ where } t \geq 2^{\frac{n-k}{n}} \text{ and } P \varepsilon \Gamma_k.$$

Further write

$$C = C_1 \cup C_2 \cup \ldots \cup C_n$$

for the union of $C_1, C_2, \ldots, C_n$, and

$$K = \mathfrak{R}_n - C$$

for the set of all points in $\mathfrak{R}_n$ which do not belong to C.

Although K is not a bounded set, it is of the finite type. For the lattice Λ_0 consisting of the points

$$(2g_1, 2g_2, \ldots, 2g_{n-1}, g_n),$$

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*) This article has been sent to J. G. VAN DER CORPUT on February 12, 1949.

1) C. A. ROGERS, The product of the minima and the determinant of a set. These Proceedings **52**, 256—263 (1949).

2) $gcd(g_1, g_2, \ldots, g_k)$ means the greatest common divisor of $g_1, g_2, \ldots, g_k$, and similarly in other cases.

where $g_1, g_2, \ldots, g_n$ run over all integers, is evidently K-admissible, and so

$$\triangle(K) \leqslant d(\Lambda) = 2^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Our aim is to find the exact value of $\triangle(K)$.

2) The origin $O = (0, 0, \ldots, 0)$ is an inner point of K, and K is of the finite type; therefore [3] K possesses at least one critical lattice, the lattice Λ say. By (1),

$$d(\Lambda) \leqslant 2^{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For $k = 1, 2, \ldots, n$, let Π_k be the parallelepiped

$$|x_h| \begin{cases} < 1 \text{ if } 1 \leqslant h \leqslant n, \; h \neq k; \\ \leqslant 2^{n-1} \text{ if } h = k. \end{cases}$$

By (2) and by MINKOWSKI's theorem on linear forms, each parallelepiped Π_k contains a point $Q_k \neq O$ of Λ. Since Λ is K-admissible, and from the definition of K, this point belongs to C; hence only the k-th coordinate of Q_k, η_k say, is different from zero and may be assumed positive:

$$Q_k = (0, \ldots, \eta_k, \ldots, 0), \qquad \text{where } \eta_k > 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The point

$$Q = Q_1 + Q_2 + \ldots + Q_n = (\eta_1, \eta_2, \ldots, \eta_n)$$

also belongs to Λ and therefore to C. Since $\eta_n > 0$, Q necessarily lies in C_n. From the definition of this set, there exist then a positive number η and n positive integers $q_1, q_2, \ldots, q_n$ such that

$$\eta_k = \eta\, q_k \qquad (k = 1, 2, \ldots, n). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

3) The n lattice points

$$Q_1, Q_2, \ldots, Q_n$$

do not necessarily form a basis of Λ; they are, however, linearly independent, and so they generate a sublattice of Λ. Hence there exists a fixed positive integer, q say, such that every point P of Λ can be written in the form

$$P = \frac{1}{q}\{p_1 Q_1 + p_2 Q_2 + \ldots + p_n Q_n\} = \left(\frac{\eta}{q} p_1 q_1, \frac{\eta}{q} p_2 q_2, \ldots, \frac{\eta}{q} p_n q_n\right)$$

with integral coefficients $p_1, p_2, \ldots, p_n$ depending on P. For shortness, put

$$\xi = \frac{\eta}{q}, \text{ so that } \xi > 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By MINKOWSKI's method of reduction [4], we can now select a basis

$$P_1, P_2, \ldots, P_n$$

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[3]) See my paper, *On the critical lattices of an arbitrary point set*, Canadian Journal of Mathematics, I (1949), 78—87.

[4]) Geometrie der Zahlen (1910), § 46.

of Λ such that each basis point P_k, where $k = 1, 2, \ldots, n$, is a linear combination of $Q_1, Q_2, \ldots, Q_k$, hence of the form

$$P_k = (\xi p_{k1}, \xi p_{k2}, \ldots, \xi p_{kk}, 0, \ldots, 0) \quad \ldots \ldots \quad (6)$$

where

$$p_{k1}, p_{k2}, \ldots, p_{kk} \text{ are integers, and } p_{kk} > 0 \quad \ldots \ldots \quad (7)$$

It may, moreover, be assumed that

$$0 \leqslant p_{kl} < p_{ll} \text{ for all pairs of indices } k, l \text{ satisfying } 1 \leqslant k < l \leqslant n. \quad (8)$$

4) **Lemma:** *Let*

$$L_h(x) = \sum_{k=1}^{n} a_{hk} x_k \qquad (h = 1, 2, \ldots, m)$$

be m linear forms in n variables $x_1, x_2, \ldots, x_n$, with integral coefficients a_{hk} not all zero. Denote by

$$a = gcd\, a_{hk}$$

the greatest common divisor of these coefficients, and by

$$L(x) = gcd\, L_h(x)$$

the greatest common divisor of the numbers $L_h(x)$, where $h = 1, 2, \ldots, m$. Then there exist integers $x_1, x_2, \ldots, x_n$ such that

$$L(x) = a.$$

P r o o f: By the theory of elementary divisors [5]), two integral unimodular square matrices

$$(b_{gh}) \text{ and } (c_{kl})$$

of m^2 and n^2 elements, respectively, can be found such that the product matrix

$$(b_{gh})\ (a_{hk})\ (c_{kl}), \qquad = (d_{gl}) \text{ say,}$$

of mn elements is a diagonal matrix, viz.

$$d_{gl} = 0 \text{ if } g \neq l.$$

Put

$$r = \min(m, n)$$

and

$$x_k = \sum_{l=1}^{n} c_{kl} x'_l, \qquad L_g(x') = \sum_{h=1}^{m} b_{gh} L_h(x),$$

so that

$$L_g(x') = \begin{cases} d_{gg} x'_g & \text{if } g \leqslant r, \\ 0 & \text{if } g > r. \end{cases}$$

Then evidently

$$a = gcd\,(d_{11}, d_{22}, \ldots, d_{rr})$$

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[5]) See e.g. B. L. VAN DER WAERDEN, Moderne Algebra, Vol. 2 (1931), § 106.

and

$$L(x) = gcd\, L_g(x') = gcd\,(d_{11} x_1', d_{22} x_2', \ldots, d_{rr} x_r'),$$

and the assertion follows on putting

$$x_1' = x_2' = \ldots = x_r' = 1.$$

5) Every point P of Λ can be written as

$$P = x_1 P_1 + x_2 P_2 + \ldots + x_n P_n$$

with integral coefficients $x_1, x_2, \ldots, x_n$. Therefore P has the coordinates

$$P = (\xi L_1(x), \xi L_2(x), \ldots, \xi L_n(x)), \quad \ldots \quad (9)$$

where, for shortness,

$$L_h(x) = \sum_{g=h}^{n} p_{gh} x_g \quad (h = 1, 2, \ldots, n). \quad \ldots \quad (10)$$

Let now d_k, for $k = 1, 2, \ldots, n$, be the greatest common divisor of the coefficients

$$p_{gh} \text{ with } 1 \leq h \leq g \leq k.$$

From this definition, it is obvious that

$$d_k \text{ is divisible by } d_{k+1} \text{ for } k = 1, 2, \ldots, n-1. \quad \ldots \quad (11)$$

Since the matrix of the n forms $L_1(x), L_2(x), \ldots, L_n(x)$ is triangular, d_k may also be defined as the greatest common divisor of the coefficients of

$$x_1, x_2, \ldots, x_k$$

in the forms

$$L_1(x), L_2(x), \ldots, L_k(x).$$

It follows therefore, for $k = 1, 2, \ldots, n$, from the lemma in 4) that there exist integers

$$x_{k1}, x_{k2}, \ldots, x_{kk}$$

not all zero such that the greatest common divisor of the k numbers

$$g_{hk} = \sum_{g=h}^{k} p_{gh} x_{kg} \quad (h = 1, 2, \ldots, k)$$

is equal to d_k.

The point

$$R_k = x_{k1} P_1 + x_{k2} P_2 + \ldots + x_{kk} P_k \neq O \quad \ldots \quad (12)$$

belongs to Λ and has the coordinates

$$R_k = (\xi g_{1k}, \xi g_{2k}, \ldots, \xi g_{kk}, 0, \ldots, 0) \quad \ldots \quad (13)$$

which are not all zero and satisfy the equation

$$gcd\,(g_{1k}, g_{2k}, \ldots, g_{kk}) = d_k. \quad \ldots \quad (14)$$

Since R_k is not an inner point of K, it belongs to one of the sets $C_1, C_2, \ldots, C_k$. We conclude therefore, from the definition of these sets, that

$$\xi d_k \geqslant 2^{\frac{n-k}{n}} \quad (k = 1, 2, \ldots, n). \quad \ldots \quad (15)$$

6) Next let ζ be the positive real number for which

$$\zeta \min_{k=1,2,\ldots,n} 2^{-\frac{n-k}{n}} d_k = 1, \text{ whence } 0 < \zeta \leqq \xi. \quad . \quad . \quad . \quad (16)$$

There is then an index $\varkappa$ with $1 \leq \varkappa \leq n$ such that

$$\zeta d_k \begin{cases} \geqq 2^{\frac{n-k}{n}} & \text{for } k = 1, 2, \ldots, n, \\ = 2^{\frac{n-\varkappa}{n}} & \text{for } k = \varkappa. \end{cases} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From these formulae (17):

$$d_k \geqq \zeta^{-1} \cdot 2^{\frac{n-k}{n}} = 2^{\frac{\varkappa-k}{n}} d_\varkappa \qquad (k = 1, 2, \ldots, n).$$

Hence, if $k < \varkappa$, then

$$d_k \geqq 2^{\frac{1}{n}} d_\varkappa,$$

whence, by (11),

$$d_k \geqq 2 d_\varkappa \qquad \text{for } k = 1, 2, \ldots, \varkappa - 1 \, . \quad . \quad . \quad . \quad . \quad (18)$$

If, however, $k \geq \varkappa$, then

$$d_k \geqq 2^{\frac{\varkappa-n}{n}} d_\varkappa > \tfrac{1}{2} d_\varkappa$$

and (11) implies now that

$$d_k = d_\varkappa \qquad \text{for } k = \varkappa, \varkappa + 1, \ldots, n. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

On combining (18) and (19), we obtain the further inequality,

$$\xi^n d_1 d_2 \ldots d_n \geqq \zeta^n d_1 d_2 \ldots d_n \geqq 2^{\varkappa-1} (\zeta d_\varkappa)^n = 2^{n-1}. \quad . \quad . \quad (20)$$

7) The critical lattice Λ we have been considering, has the basis $P_1, P_2, \ldots, P_n$ of the form (6). Its determinant is therefore

$$d(\Lambda) = \xi^n p_{11} p_{22} \ldots p_{nn}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

since all factors on the right-hand side of this equation are positive. From the definition of d_k,

$$p_{kk} \text{ is divisible by } d_k \qquad \text{for } k = 1, 2, \ldots, n. \quad . \quad . \quad . \quad (22)$$

Hence by (20) and (21),

$$d(\Lambda) \geqq \xi^n d_1 d_2 \ldots d_n \geqq 2^{n-1}$$

whence

$$\Delta(K) \geqq 2^{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The same right-hand side was, by (1), also a lower bound of $\Delta(K)$; hence the final result

$$\Delta(K) = 2^{n-1} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

is obtained.

8) By means of the last formulae, all critical lattices of K can be obtained as follows.

It is clear, from the previous discussion, that to any critical lattice Λ, there is a unique index $\varkappa$ with $1 \leq \varkappa \leq n$ such that

$$d_k = \begin{cases} 2\, d_\varkappa & \text{for } k = 1, 2, \ldots, \varkappa - 1, \\ d_\varkappa & \text{for } k = \varkappa, \varkappa + 1, \ldots, n, \end{cases} \quad \ldots \ldots \quad (24)$$

and that further

$$\zeta = \xi, \quad \ldots \ldots \quad (25)$$

$$p_{11} = d_1,\ p_{22} = d_2, \ldots, p_{nn} = d_n; \quad \ldots \ldots \quad (26)$$

for otherwise $d(\Lambda)$ would be larger than 2^{n-1}. Since we may, if necessary, replace ξ by $d_\varkappa \xi$, there is no loss of generality in assuming that

$$d_\varkappa = 1, \quad \ldots \ldots \quad (27)$$

whence, by (17):

$$\xi = 2^{\frac{n-\varkappa}{n}}. \quad \ldots \ldots \quad (28)$$

The basis points $P_1, P_2, \ldots, P_n$ become,

$$P_k = \left(2^{\frac{n-\varkappa}{n}} p_{k1}, 2^{\frac{n-\varkappa}{n}} p_{k2}, \ldots, 2^{\frac{n-\varkappa}{n}} p_{kk}, 0, \ldots, 0\right)$$

with integral p_{kl}. By (7), (8), and (24)—(28), moreover

$$p_{kk} = \begin{cases} 2 & \text{if } k = 1, 2, \ldots, \varkappa - 1, \\ 1 & \text{if } k = \varkappa, \varkappa + 1, \ldots, n, \end{cases} \quad \ldots \ldots \quad (29)$$

and

$$p_{kl} = \begin{cases} 0 & \text{if } 1 \leqslant l < k \leqslant \varkappa - 1, \\ 0 & \text{if } \varkappa \leqslant l < k \leqslant n, \\ 0 \text{ or } 1 & \text{if } \varkappa \leqslant k \leqslant n,\ 1 \leqslant l \leqslant \varkappa - 1. \end{cases} \quad \ldots \ldots \quad (30)$$

It is also clear that different choices of $\varkappa$ and of the integers p_{kl} lead to different critical lattices. Since for exactly

$$(\varkappa - 1)\,(n - \varkappa + 1)$$

coefficients p_{kl} there is the alternative $p_{kl} = 0$ or 1, there are then for each $\varkappa$ just

$$2^{(\varkappa-1)(n-\varkappa+1)}$$

different critical lattices. We find therefore, on summing over $\varkappa$, that *the total number $N(n)$ of different critical lattices of K is given by the formula*

$$N(n) = \sum_{\varkappa=1}^{n} 2^{(\varkappa-1)(n-\varkappa+1)}. \quad \ldots \ldots \quad (B)$$

Thus $N(n) = 3, 9, 33, 161, 1089, \ldots$ for $n = 2, 3, 4, 5, 6, \ldots$.

9) *We next determine the successive minima*

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

of K in the lattice Λ_1 of all points with integral coordinates.

Denote by λK, for $\lambda > 0$, the set of all points λX where X belongs to K. The first minimum λ_1 of K for Λ_1 is defined as the lower bound of all $\lambda > 0$ such that λK contains a point of Λ_1 different from O; if further $k = 2, 3, \ldots, n$, then the n-th minimum λ_k of K for Λ_1 is defined as the lower bound of all $\lambda > 0$ such that λK contains k linearly independent points of Λ_n. We find these minima as follows.

Consider an arbitrary point

$$P = (g_1, g_2, \ldots, g_n) \neq O$$

of Λ_1; here $g_1, g_2, \ldots, g_n$ are integers. Put

$$d = gcd\,(g_1, g_2, \ldots, g_n), \text{ so that } d \geq 1,$$

and assume, say, that

$$g_k \neq 0, \text{ but } g_{k+1} = \ldots = g_n = 0,$$

for some integer k with $1 \leq k \leq n$. Then P/d belongs to Γ_k, and tP belongs to C_k if and only if

$$t \geqslant 2^{\frac{n-k}{n}} d^{-1}.$$

Therefore λK, for $\lambda > 0$, contains P if, and only if,

$$\lambda > 2^{-\frac{n-k}{n}} d.$$

We deduce that if

$$\lambda \leqslant 2^{-\frac{n-1}{n}},$$

then λK contains no lattice point except O; if, however,

$$2^{-\frac{n-k}{n}} < \lambda \leqslant 2^{-\frac{n-k-1}{n}}, \quad \ldots \ldots \ldots \quad (31)$$

where $k = 1, 2, \ldots, n$, then λK contains just the points of the k sets

$$\Gamma_1, \Gamma_2, \ldots, \Gamma_k.$$

Hence, if (31) holds, then λK contains k, and not more, linearly independent points of Λ_1. The successive minima of K for Λ_1 are therefore given by the equations,

$$\lambda_k = 2^{-\frac{n-k}{n}} \qquad (k = 1, 2, \ldots, n). \quad \ldots \ldots \quad (32)$$

By (A), this implies that

$$\lambda_1 \lambda_2 \ldots \lambda_n \triangle (K) = 2^{-\sum_{k=1}^{n} \frac{n-k}{n}} 2^{n-1} = 2^{\frac{n-1}{2}} = 2^{\frac{n-1}{2}} d(\Lambda_1). \quad . \quad . \quad (C)$$

We have thus proved that *in the special case of the point set K and the*

lattice Λ_1, the sign of equality holds in ROGERS's *inequality for the successive minima of a point set* [6]).

10) The point set K is neither bounded nor a star body. It can, however, be approximated by a bounded star body of nearly the same minimum determinant and with the same successive minima, as follows.

Let ε be a small positive number. If X is any point different from O, then denote by $S_\varepsilon(X)$ the open set consisting of all points

$$tX + \varepsilon(t-1)Y$$

where t runs over all numbers with

$$t > 1,$$

and Y runs over all points of the open unit sphere

$$|Y| < 1;$$

evidently $S_\varepsilon(X)$ is a cone open towards infinity with vertex at X and axis on the line through O and X. Let further S_ε be the closed sphere of radius $1/\varepsilon$ which consists of all points Z satisfying

$$|Z| \leq 1/\varepsilon.$$

We now define K_ε as the set of all those points of K which belong to S_ε, but to none of the cones

$$S_\varepsilon\left(2^{\frac{n-k}{n}} X\right), \text{ where } X \varepsilon \Gamma_k \text{ and } k = 1, 2, \ldots, n.$$

Since only a finite number of the cones contains points of S_ε, it is clear that K_ε is a bounded star body.

Let $\lambda_1', \lambda_2', \ldots, \lambda_n'$ be the successive minima of K_ε for Λ. Since K_ε is a subset of K, necessarily

$$\lambda_k' \geqslant \lambda_k \qquad (k = 1, 2, \ldots, n).$$

We can in the present case replace these inequalities immediately by the equations

$$\lambda_k' = \lambda_k \qquad (k = 1, 2, \ldots, n) \quad \ldots \ldots \quad (33)$$

because the n boundary points

$$\left(2^{\frac{n-1}{n}}, 0, \ldots, 0\right), \left(0, 2^{\frac{n-2}{n}}, \ldots, 0\right), \ldots, (0, 0, \ldots, 1),$$

in which the successive minima of K for Λ_1 are attained, are still boundary points of K_ε provided ε is sufficiently small.

11) We further show that

$$\lim_{\varepsilon \to 0} \triangle(K_\varepsilon) = \triangle(K). \quad \ldots \ldots \quad (34)$$

[6]) See l.c. [1]).

Let this equation be false. There exists then a sequence of positive numbers

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \qquad (\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots > 0)$$

tending to zero such that

$$\lim_{r\to\infty} \Delta(\dot{K}_{\varepsilon_r})$$

exists, but is different from $\Delta(K)$. But then

$$\lim_{r\to\infty} \Delta(K_{\varepsilon_r}) < \Delta(K), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

since each K_{ε_r} is a subset of K. As a bounded star body, each K_{ε_r} possesses at least one critical lattice, Λ_r say; by the last formula, it may be assumed that

$$d(\Lambda_r) = \Delta(K_{\varepsilon_r}) \leqslant \Delta(K) \qquad (r = 1, 2, 3, \ldots).$$

Moreover, all sets K_{ε_r} contain a fixed neighbourhood of the origin O as subset. The sequence of lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is therefore bounded, and so, on possibly replacing this sequence by a suitable infinite subsequence, we may assume that the lattices Λ_r tend to a limiting lattice, Λ say. By (35),

$$d(\Lambda) = \lim_{r\to\infty} d(\Lambda_r) = \lim_{r\to\infty} \Delta(K_{\varepsilon_r}) < \Delta(K), \quad . \quad . \quad . \quad . \quad (36)$$

and therefore Λ cannot be K-admissible. Hence there exists a point $P \neq O$ of Λ which is an inner point of K. This means that P, for sufficiently small $\varepsilon > 0$, is also an inner point of K_ε.

We can now select in each lattice Λ_r a point $P_r \neq O$ such that the sequence of points

$$P_1, P_2, P_3, \ldots$$

tends to P. Hence, for any fixed sufficiently small $\varepsilon > 0$, all but a finite number of these points are inner points of K. Now, since

$$\varepsilon_r > \varepsilon_{r+1},$$

each star body K_{ε_r} is contained in all the following bodies

$$K_{\varepsilon_{r+1}}, K_{\varepsilon_{r+2}}, K_{\varepsilon_{r+3}}, \ldots.$$

Therefore, when r is sufficiently large, then the point P_r is an inner point of K_{ε_r}, contrary to the hypothesis that Λ_r is a critical, hence also an admissible lattice of K_{ε_r}. This concludes the proof of (34).

12) The two formulae (33) and (34) imply that

$$\lim_{\varepsilon\to 0} \lambda_1' \lambda_2' \ldots \lambda_n' \Delta(K_\varepsilon) = 2^{\frac{n-1}{2}} d(\Lambda_1).$$

Hence if $\delta > 0$ is an arbitrarily small number, then there exists a positive

number ε such that the successive minima $\lambda_1', \lambda_2', \ldots, \lambda_n'$ of K_ε satisfy the inequality,

$$\lambda_1' \lambda_2' \ldots \lambda_n' \triangle (K_\varepsilon) > (1-\delta)\, 2^{\frac{n-1}{2}}\, d(\Lambda_1),$$

where Λ_1 is the lattice of all points with integral coordinates.

We have therefore proved that the constant $2^{\frac{n-1}{2}}$ in ROGERS's inequality is best-possible *even for bounded star bodies*. This is very surprising as this inequality applies to general sets.

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December 15, 1948.

Postscript (May 16, 1949): In a note in the C.R. de l'Academie des Sciences (Paris), 228 (March 7, 1949), 796—797, Ch. Chabauty announces the main result of this paper, but does not give a detailed proof.

Mathematics. — *On Cardan positions for the plane motion of a rigid body.* By O. BOTTEMA. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of May 28, 1949.)

1. One of the most simple motions considered in kinematics is Cardan's elliptic motion defined by the condition that two points of the plane rigid system describe a straight line. The loci p_f and p_m of the instantaneous centre of rotation in the fixed and in the moving plane respectively are internally touching circles whose radii are in the ratio $2:1$; p_m is also the inflexion circle; each point of it describes a straight line. An arbitrary point of the plane describes an ellipse. — The question has been raised if for an arbitrary motion of the plane positions exist in which there is a contact of the n-th order with a certain elliptic motion. Contact of the n-th order means here that the orbit of each moving point has a contact of the n-th order with the curve described by the same point in the elliptic motion. It will be seen that the problem is significant for $n=3$ in view of the fact that for each position of the moving plane (some trivial cases excepted) an elliptic motion can be determined which has a contact of the second order with the given one. A position of the moving plane which has a contact of the third order with an elliptic motion will be called a *Cardan position*. The problem has been considered in an extensive but insatisfactory paper by RAUH c.s. [1]). This work was critisized by ALT [2]) in his paper on the same subject but unfortunately the latters geometrical method seems to have lead to the conclusion that a certain necessary condition for a Cardan position is also sufficient. And so also ALT's results are not correct. In the essential point of their controversy concerning the inflexion circle he is undoubtedly wrong and RAUH c.s. are right. We shall give a more analytical treatment of the problem and meet some theorems in kinematics which seem to be not generally known.

2. If x, y; X, Y are the cartesian coordinates of a point in the moving plane and in the fixed plane respectively, we have

$$\left.\begin{aligned} X &= x\cos\varphi - y\sin\varphi + a \\ Y &= x\sin\varphi + y\cos\varphi + b, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

a, b and φ being functions of the time t. As we are interested in the geometrical properties of the motion only, we take $\varphi = t$ [3]). Differentiations

[1]) RAUH, MARKS, BÜNDGENS, OTTO, Kardanbewegung und Koppelbewegung. Praktische Getriebetechnik, Heft 2, V. D. I. Verlag (1938), 1—63.

[2]) ALT, Die Kardanlagen von Getriebegliedern und die Krümmung der Polkurven, Ing. Arch. XIV (1944), 319—331.

[3]) We exclude the case $\dfrac{d\varphi}{dt} = 0$, the motion then being instantaneously a translation.

with respect to t will be denoted by primes. The position under consideration will correspond to $t=0$ and the values of the variables for this moment are indicated by the suffix 0. We have $\varphi_0 = 0$; for $t=0$ the moving and the fixed plane have the same orientation. Moreover we choose coinciding origins for this moment; thus $a_0 = b_0 = 0$.

We have

$$\left.\begin{array}{c} X' = -x\sin\varphi - y\cos\varphi + a', \quad X'' = -x\cos\varphi + y\sin\varphi + a'' \\ Y' = x\cos\varphi - y\sin\varphi + b', \quad Y'' = -x\sin\varphi - y\cos\varphi + b'' \\ X''' = x\sin\varphi + y\cos\varphi + a''' \\ Y''' = -x\cos\varphi + y\sin\varphi + b''' \end{array}\right\} . \quad (2)$$

For the instantaneous centre of rotation P, one has $X' = Y' = 0$, thus

$$x = a'\sin\varphi - b'\cos\varphi, \quad y = a'\cos\varphi + b'\sin\varphi \quad . \; . \; . \quad (3)$$

and

$$X = -b' + a, \quad Y = a' + b. \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

These equations determine p_m and p_f respectively. For $t=0$ the coordinates of P are $X = x = -b_0'$, $Y = y = a_0'$. Thus if we choose the coinciding origins for $t=0$ in the centre P, we have $a_0' = b_0' = 0$. The direction of the tangent of p_f in P is given by the ratio $-b'' + a'$, $a'' + b'$, thus for $t=0$ by $-b_0''$, a_0''; this is also the direction of the tangent of p_m in P for $t=0$. Taking this common tangent p for the X-axis we have $a_0'' = 0$. The possibilities arising from the arbitrary choise of the system of coordinates have now been exhausted and we arrive at the following conclusion: the equations for the motion (1) can be put in a canonical form for which:

$$a_0 = b_0 = a_0' = b_0' = a_0'' = 0. \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

Then we have for $t=0$ in view of (2):

$$\left.\begin{array}{llll} X = x, & X' = -y, & X'' = -x, & X''' = y + a_0''' \\ Y = y, & Y' = x, & Y'' = -y + b_0'', & Y''' = -x + b_0''' \end{array}\right\} . \quad (6)$$

Thus in order that two motions coincide for $t=0$ up to the second order it is necessary and sufficient that they have the same value b_0''; contact of the third order requires equality of b_0'', a_0''' and b_0''' for both motions.

3. The coefficient b_0'' has a geometrical meaning. The points of the plane which for $t=0$ pass through an inflexion point of their orbit satisfy the relation $X'' : Y'' = X' : Y'$, that is, in view of (6):

$$x^2 + y^2 - b_0''\, y = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Hence this equation denotes the *inflexion circle*, which touches p in P and has the *diameter* b_0''. Two motions thus coincide up to the second order if they have the same centre of rotation and the same inflexion circle. This

follows also from SAVARY's theorem or BOBILLIER's construction, which enable to determine the radius of curvature of the orbit in an arbitrary point if the centre of rotation and the inflexion circle are known.

So it is evidently possible to determine for each position of the plane an elliptic motion which has a contact of the second order. For this purpose we have only to take the motion we get when the given inflexion circle rolls within a second circle with twice the radius, P being the instantaneous point of contact [4]).

4. If an arbitrary motion has a contact of the *third* order with an elliptic one, the latter must evidently be the elliptic motion just mentioned, and we have the supplementary conditions that a_0''' and b_0''' have the same value for both motions. Therefore we determine these coefficients for the elliptic motion. We start more generally by considering the cycloidal motion obtained if a circle of radius r rolls on a circle of radius R (fig. 1).

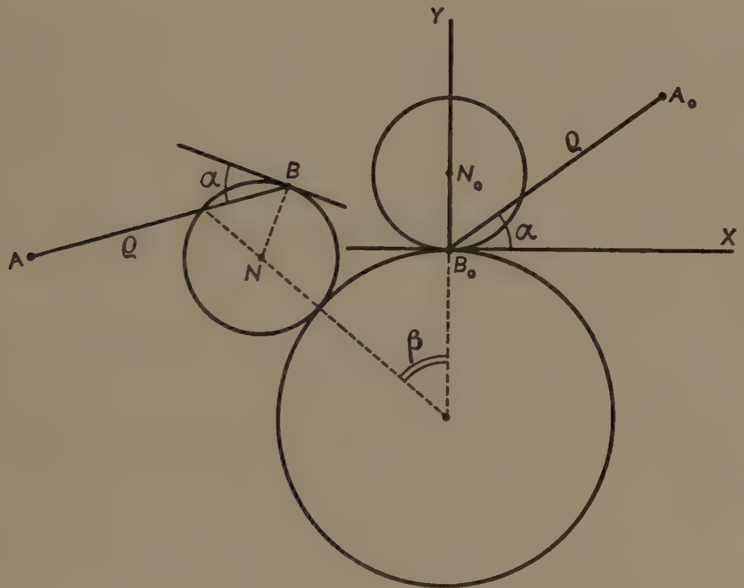

Fig. 1.

If (ϱ, α) be the polar coordinates of the point A in the moving plane and φ the angle through which this plane has rotated, then A has the following rectangular coordinates in the fixed plane:

$$X = -(R+r)\sin\beta + r\sin\varphi + \varrho\cos(\varphi+\alpha)$$
$$Y = -R + (R+r)\cos\beta - r\cos\varphi + \varrho\sin(\varphi+\alpha),$$

[4]) We exclude the case $b_0''' = 0$, the motion then having a contact of the second order with a permanent rotation.

or, since

$$\left.\begin{aligned} &\beta = \frac{r}{R+r}\varphi, \quad x = \varrho\cos\alpha, \quad y = \varrho\sin\alpha, \\ &X = x\cos\varphi - y\sin\varphi - (R+r)\sin\left(\frac{r}{R+r}\varphi\right) + r\sin\varphi \\ &Y = x\sin\varphi + y\cos\varphi - R + (R+r)\cos\left(\frac{r}{R+r}\varphi\right) - r\cos\varphi. \end{aligned}\right\} \quad (8)$$

These equations are the same as the equations (1) if

$$\left.\begin{aligned} &a = -(R+r)\sin k\varphi + r\sin\varphi \\ &b = -R + (R+r)\cos k\varphi - r\cos\varphi \\ &k = \frac{r}{R+r}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (9$$

Hence

$$\begin{aligned} a' &= -(R+r)k\cos k\varphi + r\cos\varphi, & b' &= -(R+r)k\sin k\varphi + r\sin\varphi \\ a'' &= (R+r)k^2\sin k\varphi - r\sin\varphi, & b'' &= -(R+r)k^2\cos k\varphi + r\cos\varphi \\ a''' &= (R+r)k^3\cos k\varphi - r\cos\varphi, & b''' &= (R+r)k^3\sin k\varphi - r\sin\varphi \end{aligned}$$

and in particular

$$a_0 = b_0 = a_0' = b_0' = a_0'' = 0.$$

Further

$$b_0'' = \frac{Rr}{R+r} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

which is the well-known formula for the diameter of the inflexion circle in the cycloidal motion and

$$a_0''' = -\frac{Rr(R+2r)}{(R+r)^2}, \quad b_0''' = 0. \quad . \; . \; . \; . \; . \; (11)$$

For the elliptic motion we have $R = -2r$; hence

$$a_0''' = b_0''' = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; (12)$$

So we have reached the following conclusion: *a necessary and sufficient condition for a moving plane to be in a Cardan position is that* a_0''' *and* b_0''' *are both zero.*

5. The two conditions can be associated with some properties of the state of motion. For p_f we have by (4):

$$X = -b' + a, \quad Y = a' + b.$$

Hence for $t = 0$

$$X' = b_0'', \ Y' = 0, \ X'' = -b_0''', \quad Y'' = a_0''' + b_0'',$$

and the curvature in P is

$$\frac{1}{\varrho_f} = \frac{a_0''' + b_0''}{(b_0'')^2}. \qquad (13)$$

For p_m we have by (3)

$$x = a' \sin \varphi - b' \cos \varphi, \quad y = a' \cos \varphi + b' \sin \varphi$$

and for $t = 0$

$$x' = -b_0'', \quad y' = 0, \quad x'' = -b_0''', \quad y'' = a_0''' + 2b_0''.$$

Therefore the curvature in P is

$$\frac{1}{\varrho_m} = \frac{a_0''' + 2b_0''}{(b_0'')^2}. \qquad (14)$$

From (13) and (14) it follows that in a Cardan position $(a_0''' = b_0''' = 0)$ we have $\varrho_f = 2\varrho_m$. This result was to be expected, but conversely $\varrho_f = 2\varrho_m$ implies only $a_0''' = 0$. Hence *the relation $\varrho_f = 2\varrho_m$ is a necessary, but no sufficient condition for a Cardan position.*

Here is the cause of the incorrect results in ALT's paper; moreover it is evident from our analytical treatment that the situation has to be expressed by *two* simple conditions.

6. We shall now give an other interpretation of the condition $a_0''' = 0$ an one of $b_0''' = 0$.

The inflexion circle is the locus of the points where the orbit has an inflexion point. In a point where the tangent has *four* consecutive points in common with the orbit (sometimes called *undulation point*) we have $X' : Y' = X'' : Y'' = X''' : Y'''$. Such a point lies on the inflexion circle and satisfies in view of (6):

$$a_0''' x + b_0''' y = 0. \qquad (15)$$

This line has two points of intersection with the circle: the origin P (which in general does not satisfy the conditions, X' and Y' being both zero) and the point:

$$x = \frac{-a_0''' b_0''' b_0''}{(a_0''')^2 + (b_0''')^2}, \quad y = \frac{(a_0''')^2 b_0''}{(a_0''')^2 + (b_0''')^2}. \qquad (16)$$

if a_0''' and b_0''' are not both zero. Thus there is in general *one* point with the property mentioned above. It is BALL's point [5]).

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[5]) See f.i. KRAUSE, Analysis der ebenen Bewegung (1920), p. 54.

We obviously have: if $a_0''' = 0$ (and $b_0''' \neq 0$), *then* BALL*'s point coincides with the centre of rotation P*. Hence the theorem: *if for the curvatures of p_f and p_m we have $\varrho_f = 2\varrho_m$* BALL*'s point is in the centre of rotation, and conversely.*

If, on the other hand $b_0''' = 0$ (and $a_0''' \neq 0$), then we have for the coordinates (16):

$$x = 0, \quad y = b_0''.$$

This point is on the inflexion circle and lies diametrically to the centre P; it is called the *inflexion centre* ("Wendepol"). Hence the theorem: *if* $b_0''' = 0$ (and $a_0''' \neq 0$) *then* BALL*'s point coincides with the inflexion centre.*

In a Cardan position we have $a_0''' = b_0''' = 0$; hence for this case the line (15) is undetermined and each point of the inflexion circle is a point of BALL. This was to be expected: for the elliptic motion each point of the inflexion circle describes a straight line, thus each point is an undulation point of its orbit and the same holds for a position which has a contact of the third order with an elliptic motion. ALT has been aware here of a contradiction in his arguments and has unsuccesfully tried to explain it [6]).

We have now the theorem: *A necessary and sufficient condition for a Cardan position is that each point of the inflexion circle is a point of* BALL.

7. For a point of inflexion of an orbit we have $X' : Y' = X'' : Y''$. So the locus of these points has, in view of (2), the equation

$$\left.\begin{aligned} x^2 + y^2 - x\{(a' + b'')\sin\varphi + (a'' - b')\cos\varphi\} - y\{(a' + b'')\cos\varphi - \\ - (a'' - b')\sin\varphi\} + (a'b'' - a''b') = 0 \end{aligned}\right\} \quad (17)$$

which for $t = 0$ is identical with (7) as it should be. For the diameter m of this inflexion circle we find

$$m^2 = (a' - b'')^2 + (a'' + b')^2. \quad \ldots\ldots\ldots \quad (18)$$

Hence $\dfrac{dm}{d\varphi} = 0$ when

$$(a' - b'')(a'' - b''') + (a'' + b')(a''' + b'') = 0$$

and for $t = 0$ we have

$$b_0''' = 0. \quad \ldots\ldots\ldots\ldots \quad (19$$

So we find a new interpretation of this relation: *if $b_0''' = 0$, the radius of the inflexion circle has a stationnary value, and conversely*. Thus *in a Cardan position the radius of the inflexion circle has a stationnary value.* This proposition was given, without proof, in RAUH's paper and ALT's criticism on this theorem is now seen to be unjustified. But the condition is obviously a necessary, but no sufficient one for a Cardan position.

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[6]) ALT, l.c. p. 330.

The following theorems are now evident. A necessary and sufficient condition for a Cardan position is: *the radius of the inflexion circle has a stationnary value and the ratio of the curvatures of p_f and p_m in P is* $1:2$.

If the radius of the inflexion circle has a stationnary value BALL's *point* (if determined) *coincides with the inflexion centre.*

8. The curvature k of an orbit is given by the formula

$$k = \frac{X' Y'' - X'' Y'}{(X'^2 + Y'^2)^{3/2}}.$$

Thus $\dfrac{dk}{d\varphi} = 0$ when

$$(X'^2 + Y'^2)(X' Y''' - X''' Y') - 2(X' Y'' - X'' Y')(X' X'' + Y' Y'') = 0. \quad (20)$$

From (6) we have that the locus of points with stationnary curvature for $t = 0$ is given by the equation

$$(x^2 + y^2)(a_0''' x + b_0''' y) + 2 b_0'' x (x^2 + y^2 - b_0'' y) = 0. \quad . \quad . \quad (21)$$

The locus is a cubic curve, denoted by K („Kreisungspunktkurve"). Putting $a_0''' = b_0''' = 0$ we find the theorem: *in a Cardan position the locus K of the points with stationnary curvature breaks up in the inflexion circle and the normal in the centre of rotation on the curves p_f and p_m; and conversely.* For the elliptic motion itself the theorem can easily be verified: the points of the inflexion circle describe straight lines and those of the normal are in a vertex of their orbits. If we intersect the curve K (21) with the normal $x = 0$, we get $b_0''' y = 0$. Thus on the normal lies (with the exception of P) *no* point with stationnary curvature, or *each* point of the normal has stationnary curvature. In the latter case we have $b_0''' = 0$.

9. We now consider a three-bar mechanism $\alpha AB\beta$; α and β are fixed points, A and B describe circles with the centre α and β respectively and we are interested in the motion of the bar AB. The instantaneous centre of rotation P is the point of intersection of αA and βB. Since A and B describe a circle, they must lie on the curve K. If the mechanism is in a Cardan position K is degenerated and thus A and B are either on the inflexion circle (which is impossible, their orbit being a circle) or on the above mentioned normal.

But in this case A, B and P are in a straight line and thus the mechanism

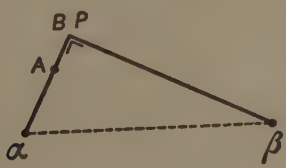

Fig. 2.

is at a deadlock. Moreover either A or B coincides with P. If B coincides with P then (from BOBILLIER's construction) βB is the tangent p of p_f and p_m and so $\angle AP\beta$ must be a right angle (fig. 2). And therefore we have the relation $\alpha\beta^2 = (\alpha A \pm AB)^2 + \beta B^2$. So in contradiction to ALT's results we can state the following theorem: *for a general three-bar mechanism no Cardan positions exist.*

10. We consider the motion of AB when A describes the circle α (R) and B the straight line l (fig. 3); the centre of rotation P is the inter-

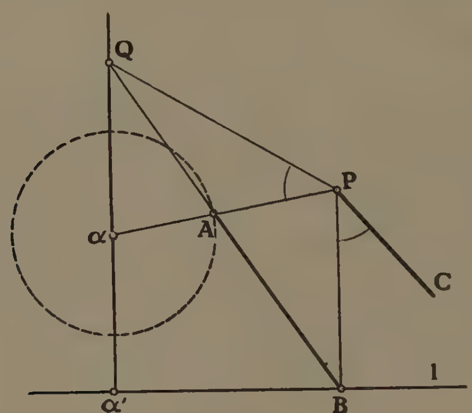

Fig. 3.

section of αA and the perpendicular in B on l; let α' be the projection of α on l, Q the intersection of AB and $\alpha\alpha'$, $\angle BPC = \angle QPA$, then from BOBILLIER's construction PC is the tangent p of p_f and p_m. For a Cardan position A must be on the normal in P, $\angle APC$ and thus $\angle BPQ$ must be right angles. If this condition is satisfied the position is a Cardan position. Indeed, on the normal lies a point (not coinciding with P) which has stationnary curvature, hence $b_0''' = 0$ and the inflexion centre is a BALL's point. But B is also such a point and therefore each point of the inflexion circle is such a point.

Now we have to investigate if the said configuration (fig. 4) is possible;

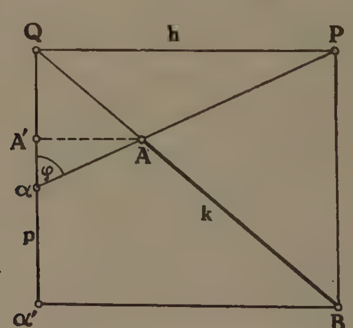

Fig. 4.

$\alpha\alpha' = p$, $\alpha A = R$ and $AB = k$ are given values, $\angle P\alpha Q = \varphi$ is an unknown angle. If $PQ = h$, A' the projection of A on αQ, then $QA' : A'A = Q\alpha' : \alpha' B$, hence

$$\frac{h \cot \varphi - R \cos \varphi}{R \sin \varphi} = \frac{h \cot \varphi + p}{h}$$

or

$$h^2 - 2hR \sin \varphi - pR \frac{\sin^2 \varphi}{\cos \varphi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Further

$$(p + R \cos \varphi)^2 + (h - R \sin \varphi)^2 = k^2$$

or

$$h^2 - 2hR \sin \varphi + 2pR \cos \varphi + p^2 + R^2 - k^2 = 0. \quad . \quad . \quad (23)$$

From (22) and (23) we have

$$pR \cos^2 \varphi + (p^2 + R^2 - k^2) \cos \varphi + pR = 0. \quad . \quad . \quad . \quad (24)$$

When this quadratic equation in $\cos \varphi$ has real roots, it has a root between -1 and $+1$, the product of the roots being 1. The discriminant of (24) is

$$D = -(p + R + k)(-p + R + k)(p - R + k)(p + R - k).$$

Since $D > 0$ when it is impossible to construct a triangle with the sides p, R and k, we reach the conclusion that the mechanism has Cardan positions if and only if p, R and k are chosen in such a way that the moving point B can *not* pass through α'.

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Geophysics. — *On true and pseudo Rayleigh waves.* By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of May 28, 1949.)

The motion of a wave caused by the reflection or refraction of a plane seismic wave at the boundary between two infinitely large media is a function of the angle of incidence; this function is of the form N/D. Without entering into the calculation of the motion in these media in a particular case it will *à priori* be evident that the roots of the equation $D = 0$ determine an important part of this motion. This part is usually called the STONELEY wave; if one of the two media vanishes it is the well-known RAYLEIGH wave [1]).

The properties of the movement in two media is extensively investigated by CAGNIARD in his book "Réflexion et réfraction des ondes séismiques progressives" [2]); the author gives also a complete description of the RAYLEIGH motion (one medium changed into vacuum).

With regard to the case in which one of the two media is a liquid CAGNIARD states that no RAYLEIGH (STONELEY) wave is possible; in view of the fact that a large part of the Earth is covered by the oceans this statement is rather important. Moreover regarding the atmosphere as a liquid layer it follows that no RAYLEIGH movement at all can exist. CAGNIARD points out that the equation $D = 0$ yields in the last case a root which is almost identical to the true RAYLEIGH solution; the motion connected with this root is therefore also almost identical to an ordinary RAYLEIGH wave.

In a previous paper [3]) I considered the propagation of disturbances through the ocean and obtained a root of the equation $D = 0$, which has been overlooked by CAGNIARD. At the end of this paper I added a note in which I commented upon the difference between the results of CAGNIARD and mine, thereby remarking that the root obtained by CAGNIARD in the case of the atmosphere is no root at all.

Prof. CAGNIARD explained to me by written communications that my rejection of this root is not only mathematically erroneous but leads to physically impossible conclusions: it is obvious that with decreasing density of the atmosphere the root of $D = 0$ has to approach the true RAYLEIGH root, which is not the case with the root I had obtained; apart from this root a second one has to exist and this is the solution given by CAGNIARD. The circumstance that the root which I found also approaches the true RAYLEIGH

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[1]) CAGNIARD calls both movements RAYLEIGH-waves, as they are essentially the same phenomenon.

[2]) GAUTHIER-VILLARS, Paris, 1939.

[3]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam., **51**, 828—835; 969—976 (1948).

root if we diminish the density of the air while letting the coefficient of compressibility unchanged (which is the case I considered) only indicates the existence of several roots. In the case: solid body-atmosphere, the root obtained by CAGNIARD is the most important and determines a pseudo RAYLEIGH wave.

The situation is therefore as follows: if one of the two media is a liquid we have two roots of $D = 0$; one of these roots is important if the density of the liquid is very small (atmosphere) — in that case the movement is the pseudo RAYLEIGH wave obtained by CAGNIARD; this movement is almost identical to the ordinary RAYLEIGH wave. In any other case this root is unimportant; the other one may cause a major surface movement (STONELEY wave) as described in my previous paper.

I am largely indebted to Prof. CAGNIARD who has in a most kind and lucid way explained this matter, thereby making this correction possible.

Zoology. — *A simple technique for the electron-microscopy of cell and tissue sections.* By L. H. BRETSCHNEIDER. (From the Zoological Laboratory, Utrecht, and the Department for Electron Microscopy, Delft.) (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of May 28, 1949.)

1. *Introductory:*

The relatively low penetration capacity of electronic rays limits electron-optical investigation to objects thinner than 10^{-3} mm. Isolated, or readily isolatable structures of vegetable or animal tissues or protozoa of such small dimensions, however, are not very numerous, so that the need to be able to make sections thinner than 10^{-3} mm was felt already when electronmicroscopy was in its first stages. The utilization of a high voltage electron microscope (made by Philips', Eindhoven, Holland), with an acceleration voltage of 100—400 kV, by the Institute for Electron Microscopy at Delft brought us nearer to the possibility to examine, when necessary, also thicker sections of, say 0,6 micron, since the penetration capacity rises with an increase in the voltage. When seeking after a suitable technique for this purpose, we found that it was possible to make plan-parallel sections through animal and vegetable tissues, of 0,6 micron, and successfully to examine these sections electronoptically. Although our aim, at present, is entirely directed towards the perfection of the technique employed, the provisional results obtained so far are already fit for publication, although we fully realize the numerous defects yet to be removed [1]).

2. *The old type of rocking microtome.*

All modern microtomes for light-microscopic investigation are adjusted to a thickness of the sections of 1 micron and multiples of this. About 60 years ago, the micron scale did not play such a dominating role, some microtomes not even having a scale division at all. This applies, for instance, to the earliest model of the "rocking microtome" made by the Cambridge Instrument Co. (London and Cambridge). In the calculation of the thickness of a section obtained with such a microtome on shifting one cog of the wheel, this thickness proved to be significantly less than 1 micron, namely, 0.59 micron. If we take care that certain conditions are fulfilled during embedding and sectioning of the object, we are able to make, with this microtome, serial sections of $\pm$ 0.6 μ, which, already at an

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[1]) During printing of this manuscript we found a publication of D. C. PEASE and R. F. BAKER about "Sectioning techniques for Electron Microscopy using a conventional microtome", in Proc. of the Soc. f. exper. Biology and Medicine, Vol. 67, 1948.

acceleration voltage of 110 kV, are thin enough for electron-microscopic examination.

The rocking microtome was originally designed by Horace Darwin, and constructed for the first time in 1885. It is probable that our abovementioned instrument is one of this earliest type. The microtome was developed from the older "Coldwell Automatic Microtome", and is characterized by the simplicity of its construction and manipulation. It owes the possibility of so minimal a shifting of the object towards the cutting blade to its typical transference of the screw movement to a lever, on which a hinge is fitted that turns a second lever carrying the object-holder. (See Fig. 1 and 2.) The nut *b* is placed in a bore in the longer extremity of the lever *a*; the spindle *c*, in this nut, moves the lever in an upward direction. This movement is communicated to the axle *d* of the lever *e* holding the object, which axle rests on the short extremity of the lever *a*. This axle describes a small sector of a circle, and thereby displaces the turning point of the object-holder *e*, and so moves it bodily towards the cutting blade *f*. The two axles *d* and *g* rest on the beam *h*, while each lever separately is held in place by a powerful steel spring *i* and *k*. The up-and-down movement of the object-holder, by which the object is drawn through the cutting blade, is effected by a cord running over a pulley towards the manipulation handle *l*. When the handle is pulled forward, the object-holder, against the tension of the spring *i*, is moved upwards, while the cogwheel is, simultaneously, and automatically, turned to the left. Only when the tension of the spring is eased, causing the manipulation handle to move backwards, does the object move through the blade.

Owing to the peculiar turning point of the lever of the object-holder, the section describes a line which is part on the surface of a cylinder. In view of the smallness of the object, e.g. 2 mm, and the length of the lever, the curvature is extremely slight. Since the cogwheel of our microtome has 260 cogs, while 6.5 revolutions — i.e. 1690 cogs — are necessary to shift the object 1 mm towards the blade, it follows that each section has a thickness of 0.59 micron. In our ancient instrument the automatic shifting of the cogwheel does not yet possess any scale-division at all; evidently it was adjusted by guesswork. In further developing the rocking-microtome into the modern model (List No. 184 of the Cambridge Instrument Co. Ltd.) the number of cogs and revolutions has been arranged in such a way that the smallest shift produces sections of 2 micron, so that this type is no longer suitable for our purpose. We have accordingly approached the firm in question, and very much hope that our suggestions and proposed improvements will lead to the construction of a microtome suitable for this purpose [2]).

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[2]) During printing of this manuscript the Cambridge Instrument Company communicated to us that building of this instrument is progressing and probably will be ready in a short time.

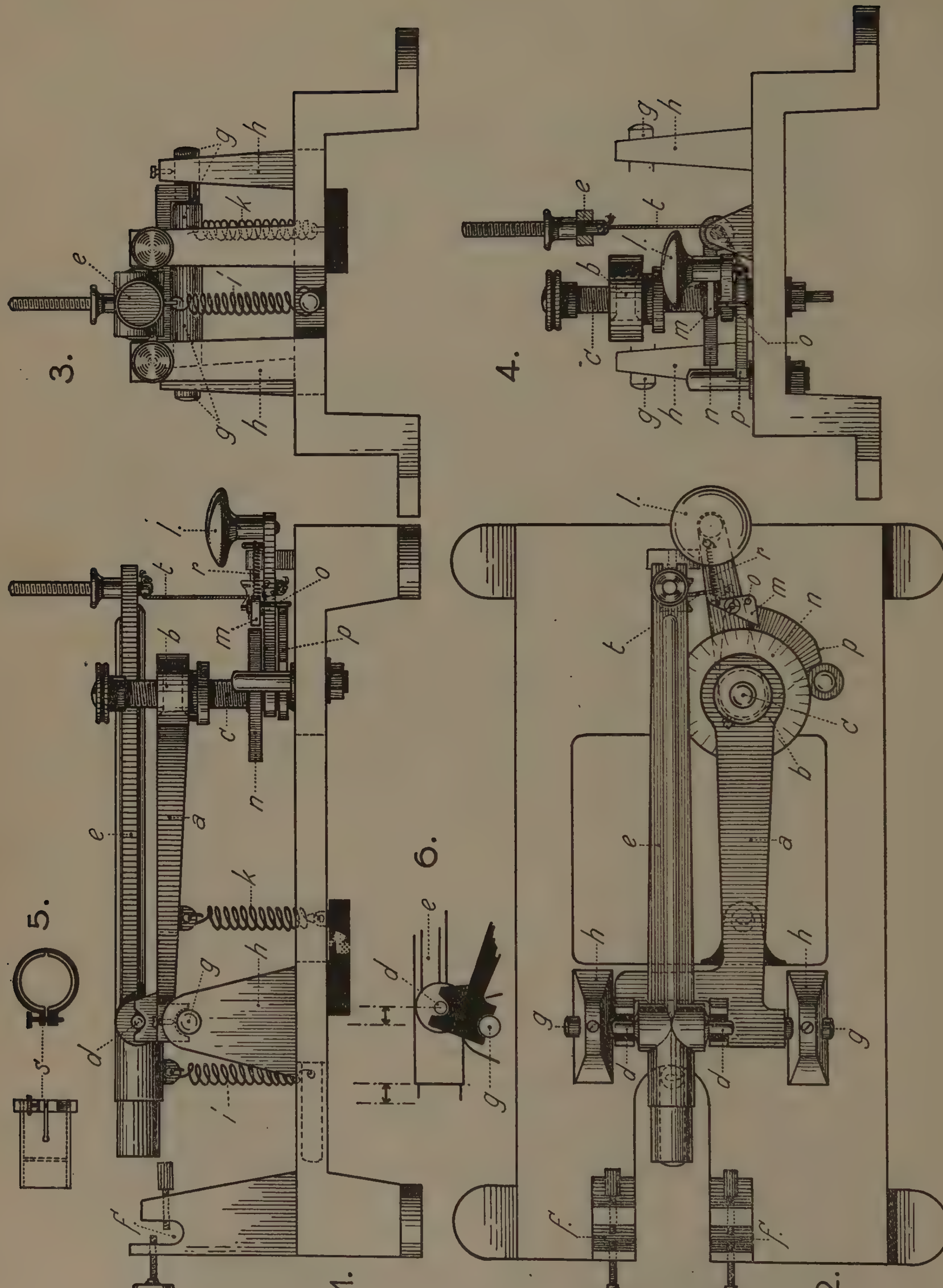


Fig. 1.

3. *The cutting blade.*

In making sections of such extreme thinness, special demands are made in regard to the properties of the cutting blade. Both the radius of curvature of the blade-point and the cutting angle should be as small as possible. The cutting edge of the blade must be polished sufficiently well to reduce to a minimum any flaws in the grinding. Although the objects (smaller than 2 mm), and the electron-microscopic field of view (less than 70 micron) are small and require only narrow blade-parts, we have, up to the present, made our sections with the ordinary, current type of microtome blades. We use the 6 cm long, concave Jung knives, which are first ground on a hone, and afterward polished on leather. We are at present seeking after a means to use Gilette blades, which more closely approach the ideal i.e. that the cutting edge passes gradually into the flat surface of the blade.

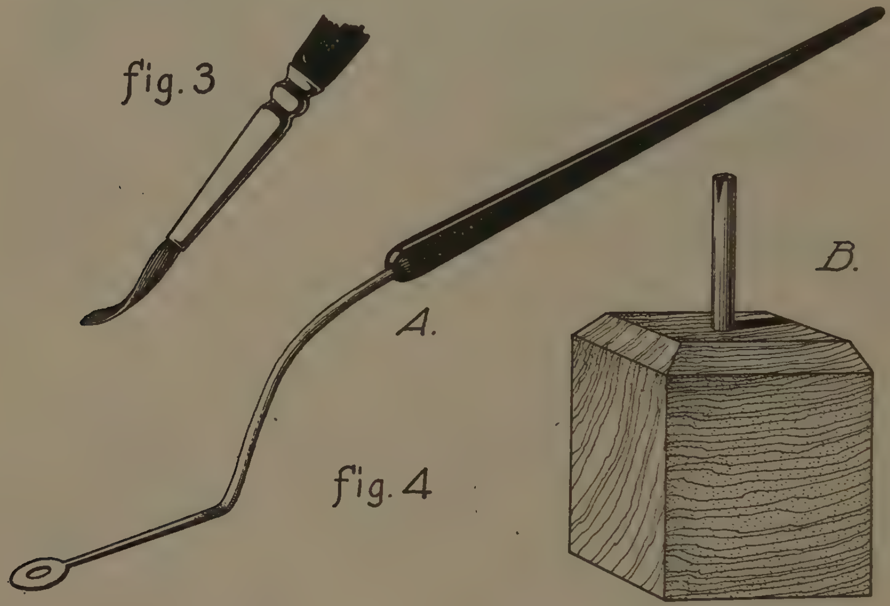
4. *Embedding, cutting and mounting of the sections.*

At ordinary room temperature, sections of such thinness, made from objects embedded in ordinary paraffin are driven up into narrow, useless strips. To prevent this, we sought after an embedding medium with a higher melting-point and a firmer consistency. For this purpose we found most useful a mixture of 2 parts yellow beeswax and one part paraffin (melting-point 72° C). The mixture is filtered a few times through a hardened filter, and kept liquid in the thermostat. Notwithstanding this raising of the melting-point, it is necessary to lower the room temperature to $\pm$ 10° C in order to prevent crushing of the sections. We have sectioned at 8° C at lowest, and up to 14° C as highest limit. Outside this temperature range the sections are no longer usable. We treat protozoa or small particles of tissue, from fixing to embedding, in centrifugal tubes. For embedding in wax-paraffin, which requires a small block not larger than 2 mm along the edge, we fold a container of thick tin-foil over a conical metal mould. On the outside, the folds are covered with a thick solution of celluloid in acetone and so closed. Objects and embedding medium are poured into this container, in the thermostat, and after sedimentation solidified in cold water. After this the container can be removed by just folding it open.

The rise in temperature of the object, placed in the usual way on to the object-holder with the aid of a heated spatula, is eliminated again by cooling in running water.

The blocks are, at most, 2 mm long and wide, and also flat in order to obviate their "giving way" and prevent any elasticity when the object is drawn through the blade. We cut small ribbons of 2—3 successive sections, which we then remove from the blade with the left hand, with the aid of a flat "water-colour" (painter's) brush. The handling of sections of such thinness requires some care. The hairs of the brush are cut level at the top. The brush is then dipped into 2 % solution of celloidin in ether-

ethanol; while drying, it is given a flat, somewhat curved form by pressing it between two fingers (fig. 3). The celloidin keeps the hairs together without making them less supple. With this brush, the sections are removed from the blade, and placed in rows into PETRI's dishes, in which they can



be kept dustfree and cool for a long time. While cutting, we from time to time make a section about 4—5 μ thick which, after staining, serves as light-microscopic control-object. During the cutting process and subsequent mounting of the sections, we successfully avail ourselves of the wellknown magnifying spectacles, which facilitate both the examination of the sections and their accurate mounting on the pellicle. In order to be able to section at a higher room temperature, we had a cooling device built for the microtome (see fig. 2, no. 2) consisting of a sectionally *L*-shaped case filled with ice chopped up fine, on which the microtome is placed. A tap serves to drain off the superfluous melted water which automatically flows away through an overflow. The front part, which is touched by the hands, is insulated by ebonite. At a room-temperature of 18° C there is a fall in temperature of the entire microtome including the object and the blade, to 12° C within 4 hours, and to 10° C after 6 hours- i.e. the optimal temperature for cutting purposes. Owing to the vertical back part of the refrigerator, the instrument, and particularly the blade, are also cooled from aside.

5. *Mounting the sections on to the object-carrier.*

The mounting of the sections on to the bore of the object-carrier of the electron-microscope has lately been considerably facilitated by the 3 mm electron microscope grids produced by the firm of Kodak (Rochester).

These small pieces of copper gauze, produced by electrolytic means and gilt, have a grille consisting of 55 μ thick wires intersecting each other at right angles, and enclosing $\pm$ 300 square openings of 70 μ. First, a parlodion pellicle is stretched over the surface of this gauze, obtained in the usual way by allowing one drop 2 % parlodion, dissolved in pure amyl acetate, to spread over the surface of aq. dest. The gauze is placed on this; it is then caught from below with the aid of a ring, at the same time one pierces the surrounding part of the pellicle with a needle (fig. 4*A*), and lays it upside down over a small anvil (fig. 4*B*), so that the pellicle now lies on top of the gauze. After drying away the superfluous water with the aid of blotting paper, and drying on a heated plate, the gauze, with its covering pellicle, is ready for the mounting of the section. The narrow ribbons referred to above are taken from the PETRI's dish by means of a brush as described earlier, and dropped on to the surface of a warm water bath. This consists of a water-tank, electrically heated to 40—45° C (fig. 2, no. 6), in which is placed a dish containing aq. dest. The heating of the bees-wax-paraffin mixture causes the surface tension to stretch the sections. When this stage is reached, the gauze, with its pellice covering, is placed into the aq. dest. with the aid of a fine pair of tweezers, exactly underneath the section, which is then lifted out of the water bath. The sections are then dried in a dustless, electrically heated box (fig. 2, no. 3) at $\pm$ 50° C, and, after drying, heated for a short time in the thermostat at the temperature of the meltingpoint of the embedding medium, which causes the object to remain quite flat on the pellicle. If the sections are not to be examined at once, the gauzes may be kept, in tubes of 4 mm diameter, with a rounded bottom and closed with a plug made of cigarette paper, in an exsiccator. Before the gauze is introduced into the electron-microscope, the embedding medium is dissolved by dipping it into xylene.

Although there is a chance that some parts of the objects are covered by the wires of the gauze, the latter's relatively large area nevertheless affords so many advantages in mounting the sections that this drawback may be regarded as being compensated to some extent. Moreover, the diameter of the openings in the gauze is greater than the diameter of the field of view of the Delft electron microscope, even at the smallest enlargement of 1500 ×.

6. *The double sections method.*

Different tissues are too compact and too hard, even at such a small size as 2 mm^2, to allow of their being sectioned to 0.6 μ. In such cases the blade may first slide a few times across the block, and then cuts far too thick a section. This applies, for example, to dense connective tissue, cartilage, bone, or horny layers. We managed, however, to obtain, also from such objects, sections of 0.6 μ thickness, by first cutting 20 μ thick sections on the freezing microtome, and afterwards embedding them, via

the alcoholic series, in the bees-wax paraffin mixture, when they can be cut to 0.6 μ. The same result was obtained with softer objects embedded in paraffin (50° C), which were first cut thick and afterwards embedded in the wax-paraffin mixture.

7. *Electron-microscopic documentation and analysis of the photographs.*

It is obvious that the Delft electron microscope, with its photographic documentation on a film band, by means of which 26 pictures can be taken without changing the film-holder, allows of more extensive documentation than the technique of other electron microscopes, which have to work with separate plates. The instrument is not always adjusted sharply enough for every picture to be eligible for reproduction, although they may still be usable in the actual investigation. For this reason the number of photographs taken is on the large side, with the consequence that the question of an easily surveyable registration has arisen in our laboratory.

We make contact prints of all pictures taken, from the films. Each print is pasted separately on a special card, which also contains the necessary data. These cards are arranged in a filing system according to the subjects to which they refer. The actual enlargements are made from the best prints, in sizes of 9 $\times$ 9 cm or 18 $\times$ 18 cm; intended for both research and publication purposes. They are pasted on drawing cardboard, and placed into boxes, numbered in the order of sequence in which they appear in the recording diary. With the aid of a magnifying glass (10 $\times$), moreover, the contact prints can be examined quite well for research purposes, and, if desired, be reproduced in drawing, by using a drawing-prism. What we lack in electron-microscopy, as compared to light-microscopy, is direct, subjective observation of the object itself, a thing which greatly facilitates analysis especially when the object is drawn. We compensate this drawback by drawing the electron-microscopic pictures, either free-hand or with the aid of a drawing-prism. This mnemotechnical analysis has many advantages, because one realizes more consciously the significance, topography, etc., while drawing the object. In cases where covering or mutually intercrossing structures are not, at first, clearly distinguished, graphic reconstructions or plastic models come to our assistance. We derive the dimensions of the structure with the aid of a weakly magnifying lens (preparation lens) with a calibrated ocular micrometer, from the enlargements. Optically, this is arranged in such a way that the distance between the ocular lines is 0.2 mm, whereby any errors in reading are reduced

8. *Experiments with sections of different thicknesses.*

We have not so far observed any essential differences in the irradiatability of tissues at the voltages used by us (110 and 240 kV). Sections thicker than 0.6 micron are definitely useless. This is clear from fig. 6 of a smooth muscle of *Mytilus edulis* (mussel), of which we recorded sections

of different thicknesses, but with the same voltage and enlargement. In the case of sections of 1.2 μ we see — electron-microscopically — the muscle fibres as 3 μ-thick, dark and seemingly compact fibres, i.e. in the same form as shown by the light-microscope. Only the enveloping, thinner reticular fibrils are plainly seen. In the case of a thickness of 1 μ, we see that the muscular fibres are built up of a number (6—10) of smaller units of $\pm$ 200 mμ thickness. Only sections of 0.6 μ, however, are thin enough to render visible the composition of these secondary fibres from the 60 mμ-thick primary smooth muscle fibrils. From this picture it is also clear that, between adjoining secondary fibres, primary fibrils invariably pass from one to the other, thus ensuring their firm interconnection. The secondary fibres are embedded in a fine reticular syncytium, of which the fibrils are about 20 mμ thick.

Owing to some slight defects, due to wear and tear, in the cogwheel of our microtome, the object was shifted, in the places in question, by the distance of half instead of one cog; with the result, however that we still obtained usable sections of only 0.3 μ (see fig. 5*b*). The two sections through the ciliate *Isotricha ruminantium*, from the bovine stomach, were made from the same object. Whereas, in the case of the section of 0.6 μ thickness (fig. 5*a*), the cilium roots in the ectoplasm lie in several layers one above the other, we see, in the 0.3 μ-thick section, only 1—2 layers. The actual ectoplasm, after fixation with mercuric chloride-alcohol-acetic acid, mainly consists of microsomes of 50—70 mμ, which fill in the space between the cilium roots, and are quite plainly visible in the 0.3 μ-thick section.

In this ciliate, the ectoplasm is separated from the endoplasm by a membrane of a thickness of about 200 mμ, in which the cilia have their roots. In the 0.3 μ-thick section a few cilia are even visible in a pure cross-section.

This — quite accidental — observation of still thinner sections than those generally used by us, has led us to point out to the Cambridge Instrument Company that, in constructing a modern microtome, the possibility of using thinner sections than 0.6 μ might be borne in mind.

9. *Some of the results:*

To illustrate the possibilities of the technique described, we will cite a few of the objects examined by us, without, for the moment, going into precise details.

Electron-microscopy of tissues aims, on the one hand, at the direct clarification of already — more or less — known light-microscopic observations on the borderline of the distinguishing capacity of the light-microscope. For this reason, electron microscopy, in these cases, consciously follows in the footsteps of the results of light-microscopy, in order to find, beyond the light-microscopic borderline, fresh details, and to check up on old ones. From this it follows at once that the enlargements to be chosen

should be increased gradually as from the light-microscopic limit, so that the relation between the total picture and the details shall not be lost. In this connection, too, the thickness of the structures to be examined should be such that this interrelation shall be maintained. The drawback — adduced as argument — that the analysis of electron-microscopic photographs of thicker structures is hindered on account of the presence of several layers, one covering the other, is less significant than the advantage derived from maintenance of the interrelation between the structural details. Radiology, too, was faced with the same difficulties and overcame them.

On the other hand, electron-microscopy consciously seeks after factual material which hitherto has been completely unknown light-microscopically, with a view to establishing more universal fundamental principles in regard to the finer structure of the organic material, and the question of the ultimate size and arrangement of its component parts. Our own investigations have shown that these two directions of research go hand in hand and are mutually complementary.

A. The cilia.

Fig. 7, of *Opalina dimidiata*, a ciliate from the intestine of *Rana esculenta*, shows, among other things, the structure and implantation of the cilia. As we see from the pictures, the cilia, which measure $\pm$ 300 mμ in diameter, consist of an axial fibril of 70 mμ thickness and a cilium sheath, or outer layer, of $\pm$ 100 mμ thickness, through which runs a spiral fibril of $\pm$ 30 mμ (fig. 7*c*). Each cilium separately is inserted in the ectoplasm by a tube-shaped cilium root, the direct continuation of the cilium sheath. This is plainly shown by the rings which we find under the pellicles in the ectoplasm, in each series of cilia caught tangentially. We also found the same structure and implantation in *Paramecium*. In *Opalina*, each cilium is surrounded at the base by a ring, while between this ring and the cilium base, 8—9 ectoplasm filaments are stretched. May be the ring and the filaments serve to keep the cilia in place.

B. The nematocysts.

Fig. 8 shows longitudinal sections through the small nematocysts of the tentacles of a sea-anemone, *Corynactis viridis*. The coils of the spirally rolled nematocyst filament in the capsule are sectioned laterally, and their structure appears complicated. As its chief characteristic, the filament which has a diameter of 500 mμ, shows a torsion which gives it the shape of a deeply cut infinite screw-thread. In the wall of the filament there are two spiral bands, each 100 mμ wide, which appear in the projection on the photographs as two bands intersecting each other. Around these coils lies the actual closed capsule membrane, which is extremely thin (see fig. 8*b*). The mechanism of a nematocyst is based on the action of a temporary, higher inside pressure in this capsule, which causes the filament to be expelled. For this reason it seemed to us of interest to ascertain whether

this membrane might perhaps possess a certain electron-microscopic structure, which would be adapted to this turgescence function. It appears, indeed, from greater enlargements, that the capsule wall consists of parallel fibrils of ± 90 Å thickness, running in the longitudinal direction of the nematocyst capsule. These fibrils show a very regular striation at right angles with their direction due to the fact that denser and less dense bands alternate at equal distances. The periodicity of this striation is ± 300 Å units, and very probably represents the macromolecular arrangement of the protein chains forming the structure of this membrane.

C. *Bone structure.*

Fig. 9 shows a section through a lamella spongiosa of the vertebral bone of the bluewhale *Balaenoptera musculus*. The bone was fixed in formalin immediately after the animal was caught; it was then decalcified, and further treated according to the double section method described on p. 659. The picture was made from the outer zone of a lamella, and shows a fine three-dimensional network of collagenic fibrils oriented longitudinally. The fibrils measure between 100 and 200 mμ, while the interspaces are between 100 and 300 mμ wide. In these spaces, and around the fibrils we see spherical bodies of about 100 mμ in diameter, lying like a cloak around the fibrils. This, quite probably, is the mucoid secreted by the bone-cell and masking the collagenic fibrils. Owing to the many anastomoses between the fibrils, the interspaces also appear in the guise of a three-dimensional network, in which the calcium salts were originally deposited.

D. *Structure of cross-striated muscles.*

The pictures in fig. 10 and 11 derive from sections through the wing-muscle of a dragon-fly *Aeschna cyanea*. Here, we invariably find only a single layer of the 300 mμ-thick myofibrils in evidence. The sarcomeres, about 2—2.5 μ in length, are distinctly bounded by ring-shaped Z bands (see fig. 11). On other photographs — not reproduced here — these Z bands consist of membranes transversely intersecting the muscle syncytium. As homologues of the Q band we see here ± 1.5 μ large bodies, the sarcosomes, which surround each myofibrilla like a sheath (see fig. 11), and, in fixation, are shorter than the sarcomeres. These sarcosomes have an alveolar structure. In greater enlargements we see that each myofibril consists of a larger number of protofibrillae (see especially near the Z bands, fig. 11), which measure ± 25 mμ in diameter. In fibrils without sarcosomes we notice a transverse striation, with a periodicity of 400—500 Å units, and probably due to the actomyosin.

We also investigated the myogenesis in very young larvae of *Triton taeniatus*, because there the myoblasts in the tail are younger and less far differentiated than in the trunk which enables us to observe the genesis of the cross striation in one and the same object (see fig. 12 and 13).

In the myoblasts of the tail we see the presence of primary fibrillae with a diameter of 120 mμ which are not yet transversely striated, and are joined together in bundles. After this, parallel septa are formed in the protoplasm eventually becoming the well-known Z- and M bands, on the fibrillae and forming the links between the myofibrillae mutually. We plainly see here that the Q band is thicker than the I band, giving the impression as if, also in vertebrates, the fibrilla in the Q band is surrounded by a sheath, at any rate during the period of its genesis. During the genesis of these myofibrils we find, in the plasm of the myoblasts, spherical, 1 μ-thick sarcosomes accompanying the still non-striated fibrillae at regular distances; they may possibly be related in some way to the Q zone and its genesis.

E. Plasm Structures.

Fig. 14 shows the local structural differences, fixation and object being the same — which may occur in the plasm of part of the fertilized oocyte of *Ascaris megalocephala*. In this worm, the egg shell is formed immediately after insemination by the secretion of ascarylic acid from the oocyte; a process still taking place here. It will be seen how the coarse vacuolar plasm structure in the remaining part of the oocyte (at the bottom of the picture), at the point where the secretion is still proceeding, takes on the character of an extremely fine vacuolar structure, while the cortex of the oocyte is being broken by a liquid current. The fact that the egg membrane is not yet fully formed in this place is evident from its thinness and from the hiatus. The egg-membrane itself possesses a large number of pores from 20—40 mμ in diameter, and is built up in layers.

F. Intestinal cell.

The structure of the striated border, and the plasm zone below it, in the intestinal cell of *Ascaris*, are shown in fig. 15. The striated border consists of 5 μ-long and 80 mμ-thick filaments kept together by means of thin transverse links. They pierce the thin limiting membrane of the intestinal cell and pass into the plasmatic structure of the cortex, which latter shows an extremely dense configuration. Thereunder begins the endoplasm, which contains numerous mitochondria (Osmium fixation). These consist of filaments with a thickness of 160 mμ, whose substance, on the greater enlargements, gives the impression of consisting of fibrillae in a parallel arrangement. At the surface of these seemingly homogeneous mitochondria we find, may be as a product of condensation, numerous denser granula. The larger granula of 1—1.5 μ, strongly impregnated by OsO_4, consist of reserve fat.

It may be remarked even now, from the above concise commentary of the photographs here published, that in each of the objects investigated, there appear not only additions to what was already known light-microscopically, but also structures newly discovered by electron microscopy.

The further development of the technique described above must now be directed more especially towards an extensive investigation of the most efficient fixation and staining methods to be applied in the electron-microscopic analysis.

10. *Summary.*

(1) A description is given of a simple microtome (old model "Rocking Microtome, Cambridge"), and a fairly simple histological technique by which sections of 0.6 microns can be made through protozoa, animal and vegetable tissues. An ice-cooled installation is described, by means of which sectioning is possible also at room temperature.

(2) Electron-microscopic photographs are described, of sections of 1.2 μ, 1 μ, 0.6 μ and 0.3 μ, from which it appears that sections of 0.6 μ are already thin enough for examination with the Delft electromagnetic electron microscope (*vide* J. B. LE POOLE, *Philips Technical Review*, Vol. 9, pp. 35—46, 1947), at an emission voltage of 110 kV and with the electron microscope with high acceleration voltage for 400 kV (A. C. VAN DORSTEN, W. J. OOSTERKAMP, and J. B. LE POOLE, *Philips Technical Review*, Vol. 9, pp. 195—201, 1947), at emission voltages of 240 kV.

(3) Some results, elucidated by photographs, obtained with the following objects, are discussed to illustrate the technique described:

(*a*) Ciliary structure and implantation in *Isotricha, Opalina* and *Paramecium*;

(*b*) nematocysts in *Corynactis*;

(*c*) bone structure in *Balaenoptera*;

(*d*) wing muscle in *Aeschna*; cross-striated muscle in the *Triton* larva;

(*e*) egg membrane secretion and plasm structure of the oocyte in *Ascaris*, and

(*f*) striated border and mitochondria in the intestinal cell of *Ascaris*.

We wish to express our sincere thanks to Mr. J. B. LE POOLE, C.E. and the staff of the Department for Electron Microscopy, Delft, for their untiring and valuable cooperation. Our investigations would, moreover, not have been possible without the financial assistance of the "Netherlands Organization for Scientific Research" (Z.W.O.), to which body we hereby also tender our grateful thanks.

Fig. 2. General view of the materials.

Fig. 5. Sections of *Isotricha*; fixation with FLEMMING's fluid; thickness a) 0'6 μ and b) 0'3 μ; orig. magn. 25.000 ×; 110 KV.

Fig. 6. Sections of a muscle of *Mytilus;* fixation with BOUIN's fluid; thickness a) 1'2 μ, b) 0'9 μ and c) 0'6 μ; orig. magn. 10.000 ×; 110 KV.

Fig. 7. Sections of *Opalina* and *Paramecium*; fixation with FLEMMING's fluid; thickness 0'6 μ; a) orig. magn. 24.000 ×, b) and c) 20.000 ×; 110 KV.

Fig. 8. Sections of nematocysts of *Corynactis*; fixation with BOUIN's fluid; thickness 0'6 μ; orig. magn. a) and c) 15.000 ×, b) 30.000 ×; 110 KV.

Fig. 9. Section of bone of *Balaenoptera*; fixation with formalin; thickness 0'6 μ; orig. magn. 20.000 ×; 110 KV.

Fig. 10. Section of a muscle of *Aeschna*; fixation with BOUIN's fluid; thickness 0'6 μ; orig. magn. 16.000 ×; 240 KV.

Fig. 11. Section of a muscle of *Aeschna*; fixation with BOUIN's fluid; thickness 0'6 μ; orig. magn. 16.000 ×; 240 KV.

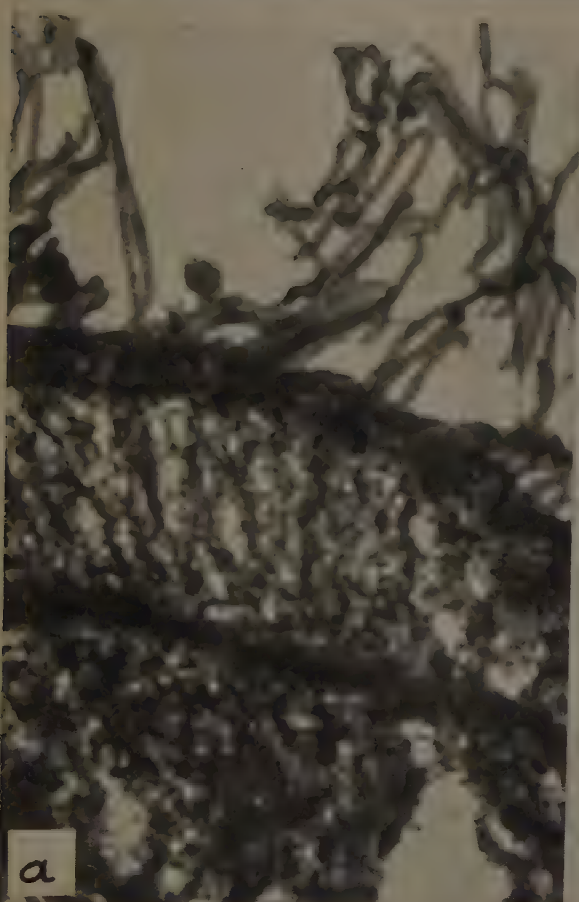
Fig. 12. Section of a myoblast of *Triton*; fixation with LENHOSSEK's fluid; thickness 0'6 μ; orig. magn. 8.000 ×; 110 KV.

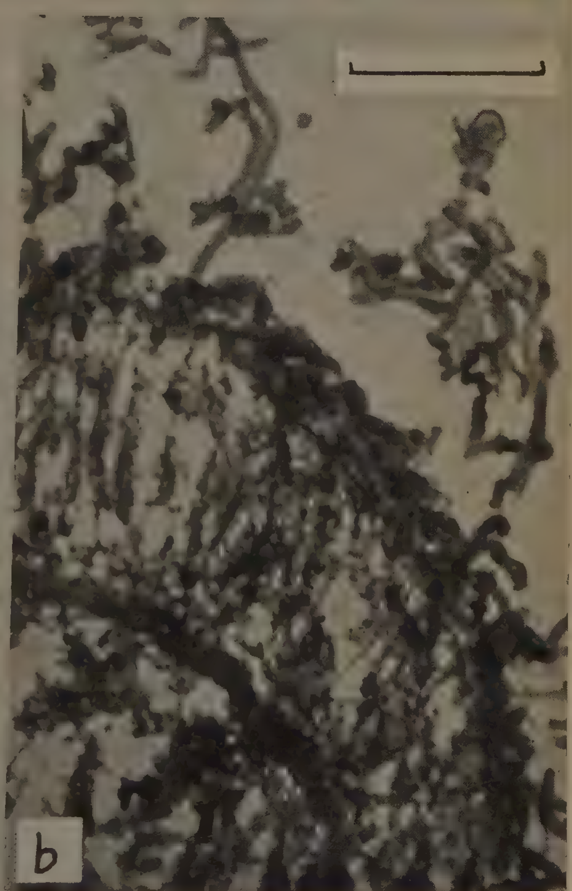
Fig. 13. Section of a myoblast of *Triton*; fixation with LENHOSSEK's fluid; thickness 0'6 μ; orig. magn. 12.000 ×; 110 KV.

Fig. 14. Section of an oocyte of *Ascaris*; fixation with CARNOY's fluid; thickness 0'6 μ; orig. magn. 12.000 ×; 110 KV.

Fig. 15. Section of an intestinal epithelial cell of *Ascaris*; fixation with CHAMPY's fluid; thickness 0'6 μ; orig. magn. 8.000 ×. 110 KV.

L. H. BRETSCHNEIDER: *A simple technique for the electron-microscopy of cell and tissue sections.*





|———| Indicates 1 μ in all pictures.

Figs. 2 and 5.

Fig. 6.

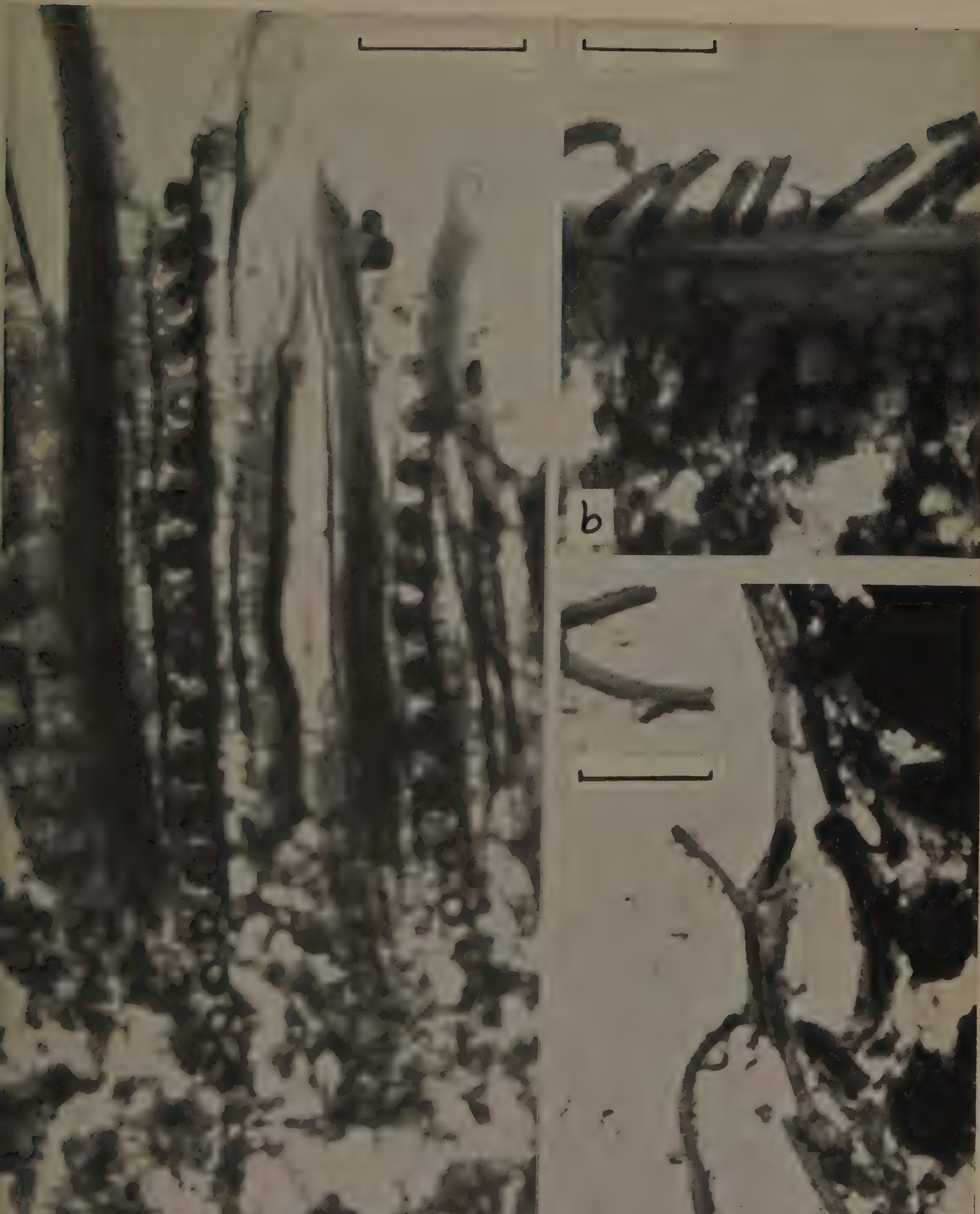


Fig. 7.

Fig. 8.

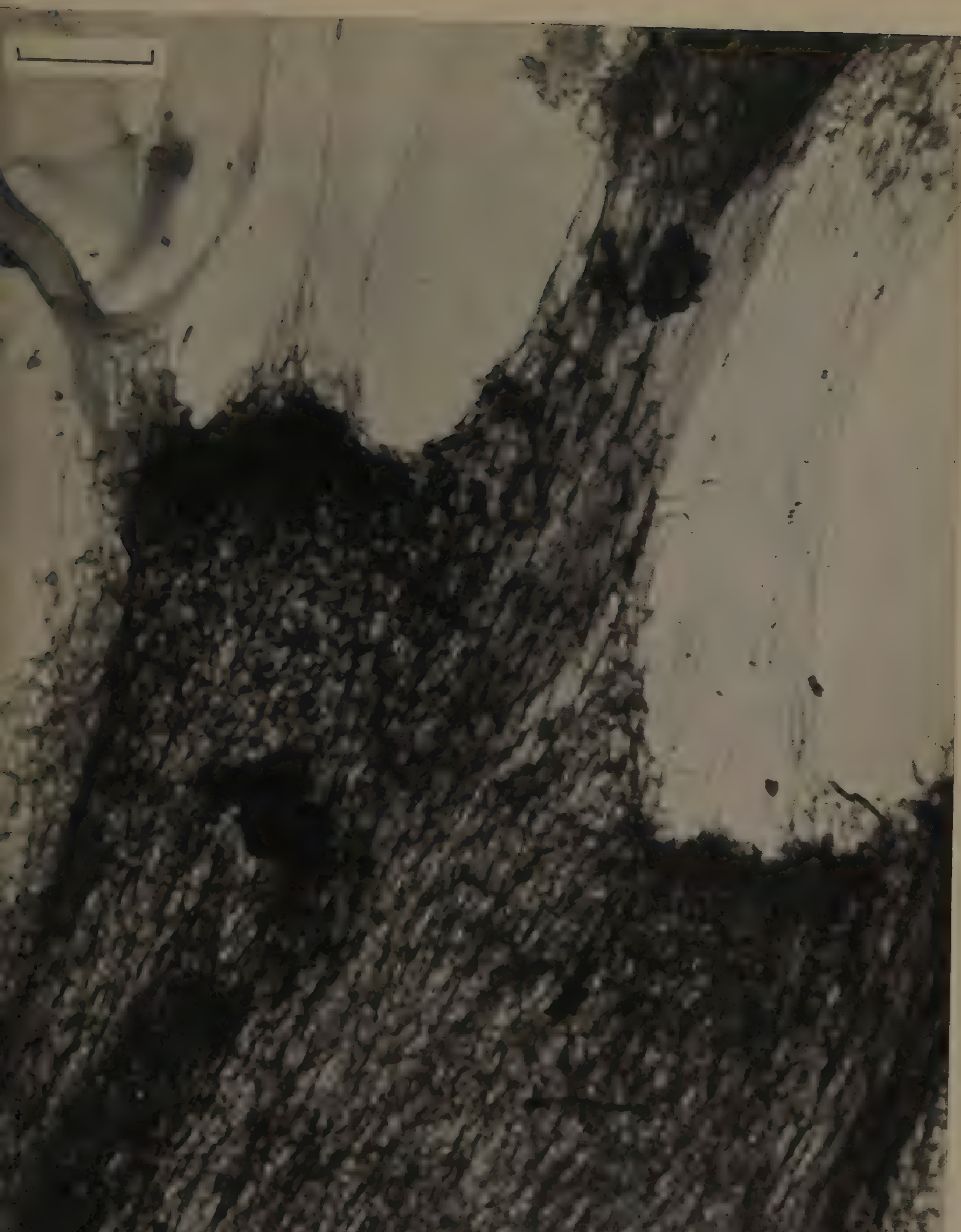


Fig. 9.

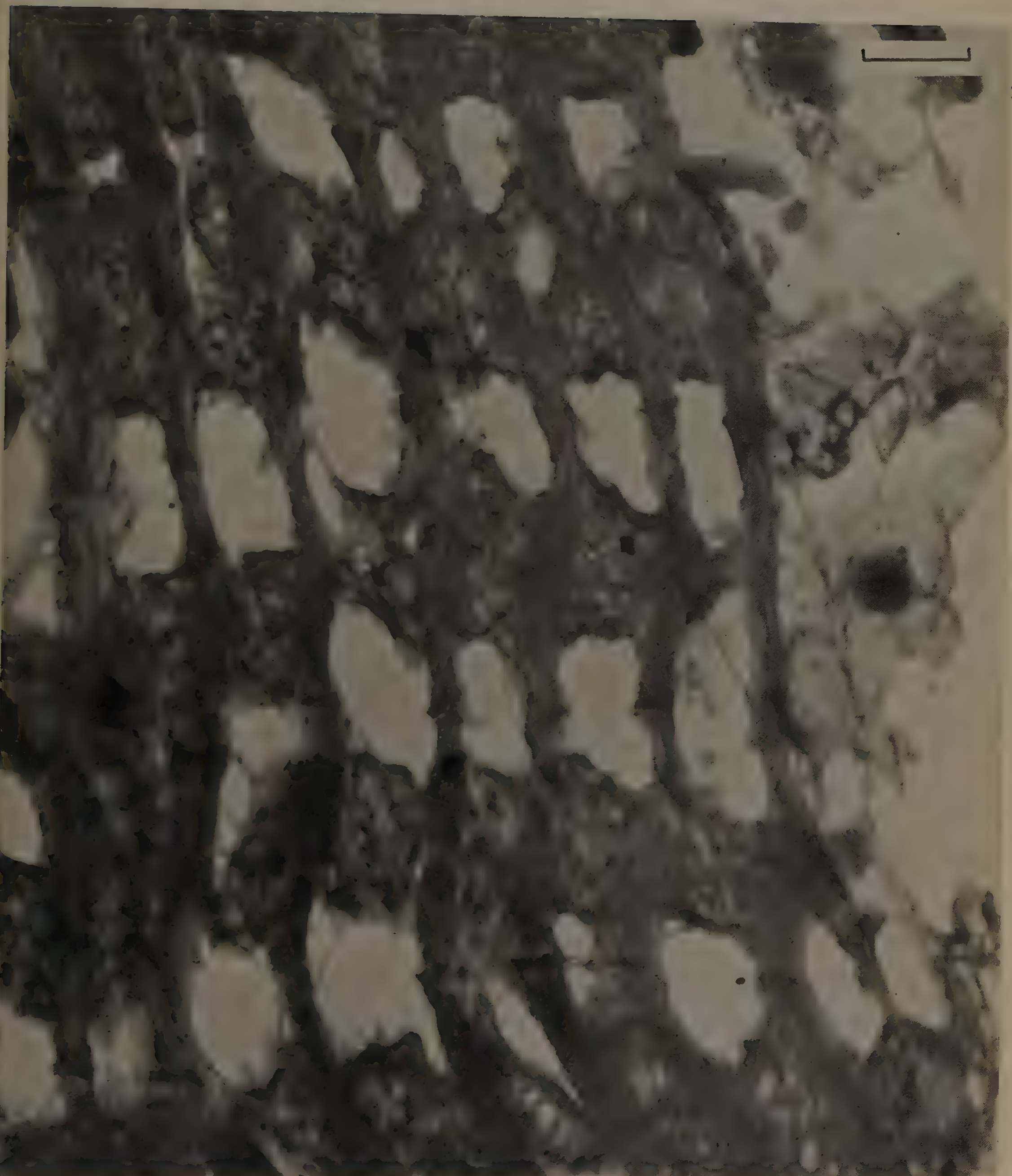


Fig. 10.

Fig. 11.

Fig. 12.

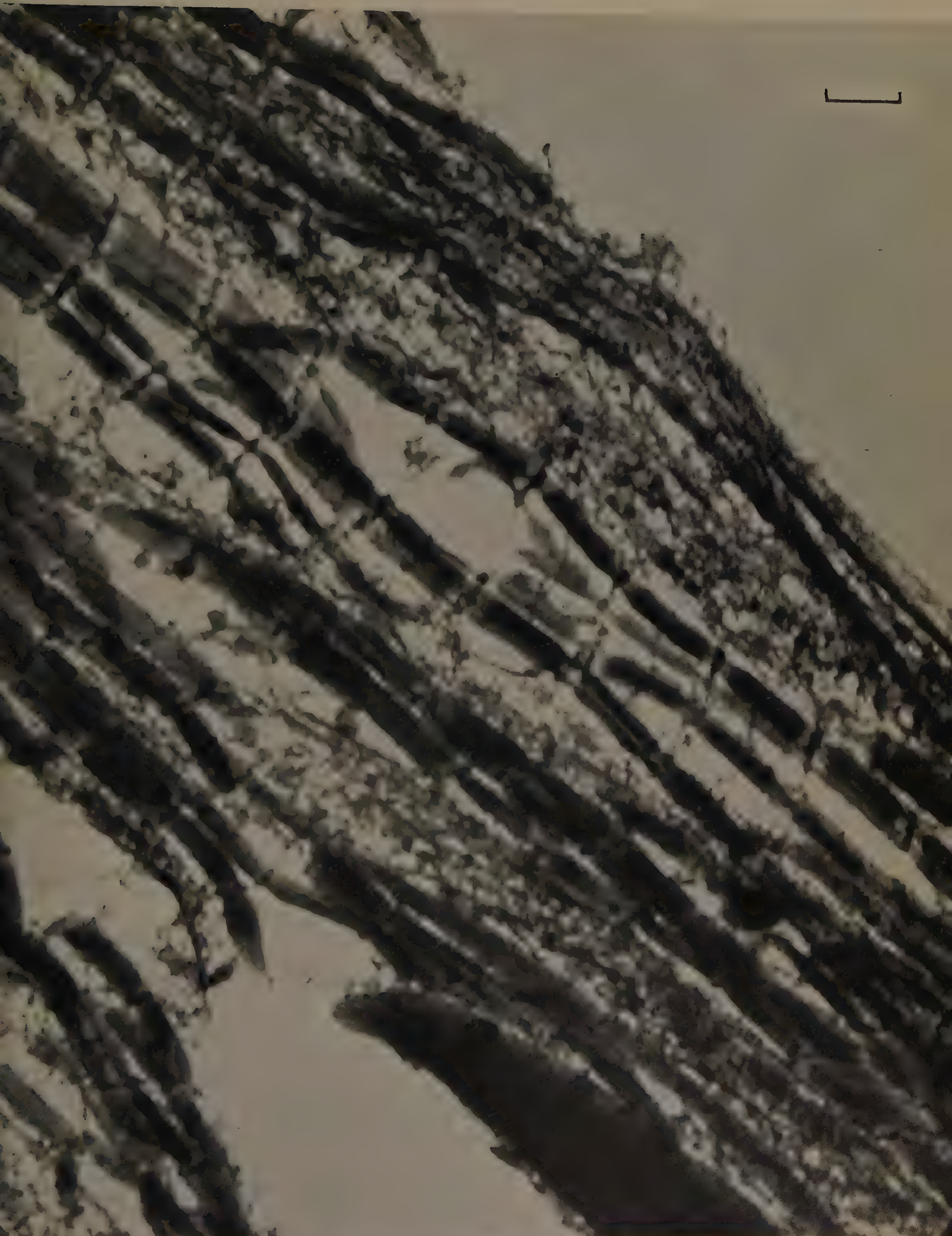


Fig. 13.

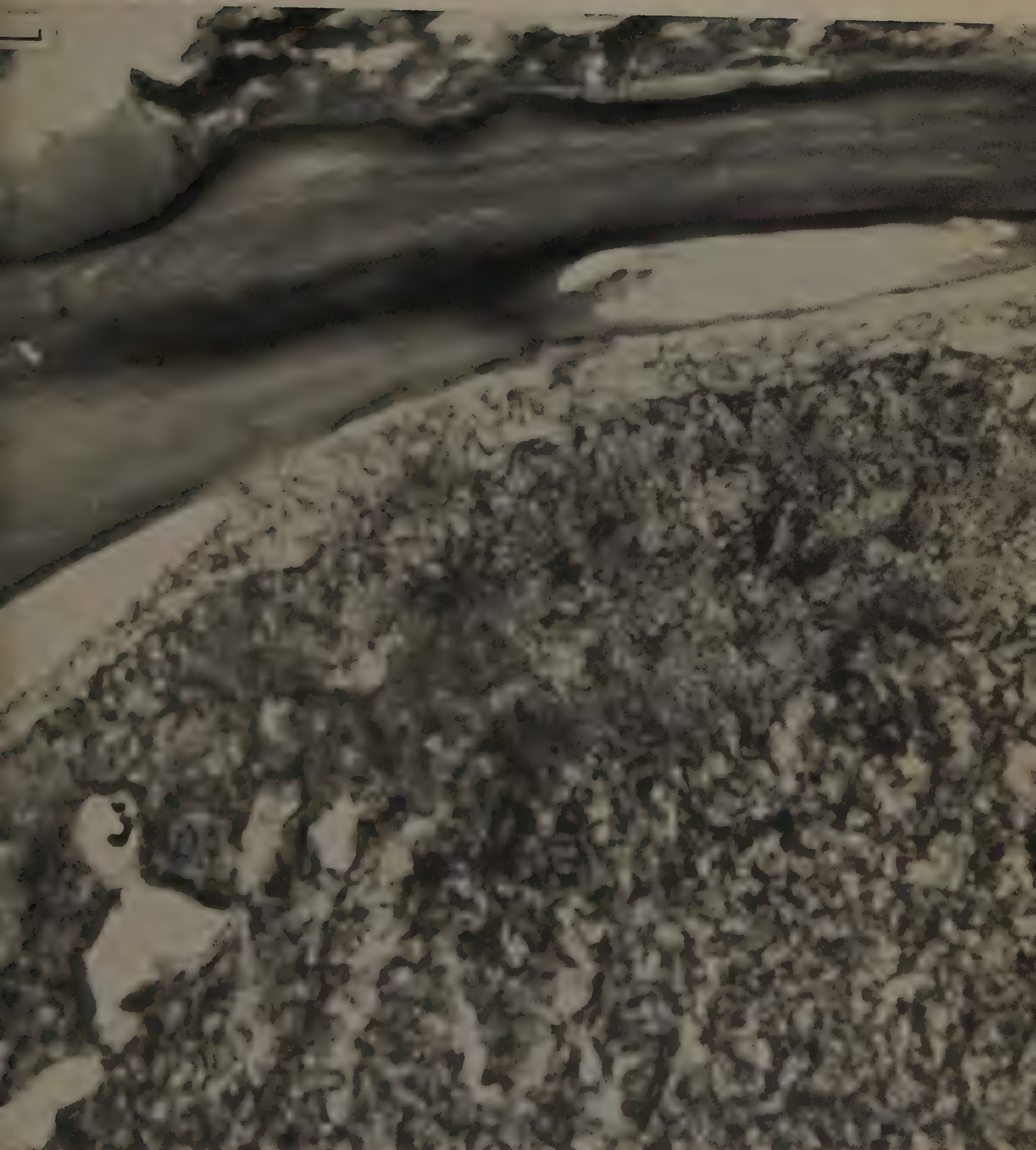


Fig. 14.

Fig. 15.

Botany. — *De hypothese voor de erfelijkheidsformules van de twee zuivere lijnen I en II van Phaseolus vulgaris op grond van kruisingsproeven.* III. By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of January 29, 1949.)

Pl. 68, F_3-1935 is het 3de voorbeeld, dat we hier bespreken van een bonenopbrengst met de formule $L_1 l_2$ $B_1 b_2$ der gemiddelde lengte en breedte. De gemiddelden komen zeer overeen met die van pl. 58 (tab. 6). De uitgangsboon voor pl. 68 is van pl. 76, F_2-1934. De eerste boon van de peul van de uitgangsboon heeft een kleine lengte en ook een kleine breedte. De 3de, laatste boon van de peul is zeer klein (klein gew. = 36 cg). De L B-indices van de eerste en de tweede boon zijn niet hoog (L B = 63.5 en 62.5). Het zijn geen kenmerkende bonen voor de formule $L_1 l_2$ $B_1 b_2$. De bonenopbrengst van pl. 76 bevat vrij veel bonen met een grote breedte en een kleine of niet zeer grote lengte.

Van de bonenopbrengst van 51 bonen van pl. 68, F_3-1935, is de grootste breedte, b = 10.9 mm, de lengte van deze boon is l = 17.0 mm, dus ook zeer groot (tab. 7). Er zijn 10 bonen met de grote breedte, b = 10.1—10.6 mm, de lengte van deze bonen is, l = 14.5—15.6 (dan volgt 15.5, dan 15.3) mm. Er zijn dus slechts 2 bonen met de grenswaarde voor zeer grote lengte (15.6 mm). Er zijn 25 bonen met de grote breedte 9.6—10.0 mm, de lengte van deze bonen is 13.3—15.6 (dan volgt 15.4) mm. Er zijn 9 bonen met de breedte, b = 9.1—9.5 mm, de lengte is, l = 13.6—14.6 mm. Ten slotte zijn er nog 6 bonen met de breedte b = 7.8 (dan volgt 8.5)—9.0 mm; de lengte is 12.6—14.4 (dan 13.7) mm. De bonenopbrengst van pl. 68 heeft voor de lengte en de breedte de formule $L_1 l_2$ $B_1 b_2$, het genotype van de uitgangsboon van pl. 76 voor pl. 68 is dus $L_1 l_2$ $B_1 b_2$.

Van pl. 68, F_3-1935, zijn in 1936 4 bonen voortgekweekt. Ze leverden de pl. 291—294, F_4-1936 (tab. 6).

Pl. 291. De uitgangsboon van pl. 68 voor pl. 291 is de boon met de grootste breedte en de grootste lengte van de bonenopbrengst van pl. 68. De 2de boon van de peul heeft niet zo'n grote breedte (b = 10.3 mm) en een veel minder grote lengte (l = 15.2 mm); daardoor een hoge L B-index (L B = 68). Bij de bonenopbrengst van pl. 291 van slechts 15 bonen staat aangetekend: „slecht, veel onrijpe peulen, bonen gevlekt, met wat rimpels". De kleine bonenopbrengst is zeer gelijkmatig samengesteld. Slechts één boon heeft de kleine breedte, b = 8.5 mm, (het is de laatste boon in de rij van de peul, l = 13.1 mm, L B = 65). Van de overige 14 bonen is de breedte, b = 9.3—10.1 mm, de lengte, l = 13.3 (dan 13.6) —14.6 (dan 14.4) mm. De L B-index is L B = 66—78 (dan volgt 73). De bonenopbrenst heeft het phaenotype $L_1 l_2$ $B_1 b_2$. Daarbij, de verschillende milieuverhoudingen van 1935 en 1936 in aanmerking nemende (blz. 584 en tab. 3), vatten we de zeer grote lengte van de uitgangsboon van pl. 68 voor pl. 291 als een niet-erfelijke variant op. Een niet-erfelijke variant dus van de form. $L_1 l_2$.

Pl. 292. De uitgangsboon van pl. 68 voor pl. 292 heeft een grote breedte en een niet zeer grote lengte; de L B-index is hoog (tab. 6). Alle 4 bonen van de peul hebben een grote breedte en een niet zeer grote lengte (van de 2de boon is l = 15.3 mm, L B = 63).

Van 6 bonen van de bonenopbrengst van pl. 292 is de breedte, b = 9.7—10.4 mm, de lengte is l = 14.0—16.1 (dan 15.8, dan 15.3) mm (z. ben.). Van 5 bonen is b = 9.3—9.5 mm, de lengte is l = 13.1—15.3 (dan 14.3) mm. L B-ind. = 62 (dan 66) —68. Van 5 bonen is de breedte b = 8.7—9.0; de lengte is l = 11.4—12.6 mm. L B-ind. = 65—72. Ook deze bonenopbrengst heeft een gelijkmatige samenstelling; de formule is overwegend $L_1\, l_2\, B_1\, b_2$. Eén peul heeft uitzonderlijk grote en zware bonen (gew. = 71—76 cg). Alle 3 bonen van deze peul zijn in 1936 voortgekweekt; bovendien is nog een boon met een grote breedte voortgekweekt. Deze 4 bonen leverden de pl. 221—224, F_4-1936 (tab. 6).

Pl. 221. F_5-1937. De uitgangsboon is van pl. 292 en heeft de form. $L_1\, l_2\, B_1\, b_2$ voor de lengte en de breedte (tab. 6). Ook de 2 overige bonen van de peul hebben deze formule. De bonenopbrengst van 32 bonen van pl. 221 bevat 9 bonen met de breedte 9.6—10.2 mm; de lengte van deze bonen is l = 14.6—16.8 (dan 15.8) mm. De L B-index is L B = 61—66. Er zijn 5 bonen langer dan 15.5 mm. Van 13 bonen is de breedte b = 9.1—9.5 mm, de lengte is l = 13.4—15.3 mm; L B-ind. = 61—69; van 7 bonen is L B = 61—65. Ten slotte zijn er 10 bonen met b = 8.3—9.0 mm; de lengte is l = 12.9—14.4 mm, en L B = 62—67.

De samenstelling van de bonenopbrengst is gelijkmatig, doch ze is niet kenmerkend voor de form. $L_1\, l_2\, B_1\, b_2$: er zijn veel bonen, die niet zeer breed zijn en vrij veel bonen met een zeer grote lengte; ook is van veel bonen de L B-index niet zeer hoog. We nemen aan, dat de uitgangsboon van pl. 292 voor pl. 221 niet de form. $L_1\, l_2\, B_1\, b_2$ in homozygote vorm of bijna homozygote vorm heeft.

Pl. 222, F_5-1937. De lengte van de uitgangsboon van pl. 292 voor pl. 222 is zeer groot (tab. 6). Van de bonenopbrengst van 30 bonen hebben 16 bonen de grote breedte, b = 9.6—11.1 (dan 10.1) mm, de lengte is l = 14.6 (dan 14.8, dan 15.6)—17.7 mm. Veertien bonen hebben dus een zeer grote lengte. De L B-index is L B = 57—65 d.i. laag. Van 9 bonen is b = 9.2—9.5 mm, l = 14.4 (dan 14.7, dan 15.2)—16.8 (dan 16.3) mm. Daarbij zijn 3 bonen met een zeer grote lengte. De L B-index is L B-57—65, dus laag. Ten slotte is van 5 bonen de breedte, b = 8.4—9.0 mm en de lengte, l = 13.6—14.9 mm. De L B-index is L B = 60—66.

De bonenopbrengst van pl. 222 is gelijkmatig; dit wijst er op, dat de formule van de uitgangsboon van pl. 292 voor pl. 222 de form. $L_1\, L_2\, B_1\, b_2$ heeft in de homozygote of bijna homozygote vorm, dus in overeenstemming met het phaenotype van de uitgangsboon.

Pl. 223, F_5-1937. Ook de uitgangsboon van pl. 292, — van dezelfde peul als die voor pl. 222 —, heeft een grote breedte en een zeer grote lengte (tab. 6).

Ook de bonenopbrengst van pl. 223 heeft een gelijkmatige samenstelling. Van de 29 bonen hebben er 13, de breedte, b = 9.6—10.4 mm, de lengte is l = 15.8 (dan 15.4)—16.8 mm. Slechts van één boon is de lengte niet zeer groot. De L B-index is 60 (dan 63)—65. Slechts van één van deze bonen is L B = 65. De L B-index is dus niet hoog. Van 5 bonen is de breedte b = 9.1—9.4 mm en de lengte, l = 14.4—15.2 (dan 14.5) mm. De L B-index is 60—64. Van 11 bonen ten slotte is b = 8.4—9.0 mm en de lengte, l = 13.4—15.6 (dan 15.1) mm; L B = 56—66 (dan 63).

Het phaenotype van de bonenopbrengst van pl. 223 sluit zeer aan bij dat van de bonenopbrengst van pl. 222, zoals ook de beide uitgangsbonen overeenkomen. Pl. 223 is ook een goed voorbeeld van een bonenopbrengst, waarvan de uitgangsboon de form. $L_1\, L_2\, B_1\, b_2$ voor de lengte en de breedte in de homozygote of bijna homozygote vorm heeft.

Pl. 224. Ook van pl. 224, F_5-1937, is de uitgangsboon van pl. 292, F_4-1936 (tab. 6). Het is de laatste boon van de peul met 3 bonen, waarvan ook de uitgangsboon voor de pl. 222 en 223 genomen zijn. De lengte en de breedte zijn iets kleiner. Van 8 bonen is de breedte, b = 9.6—10.0 mm en de lengte, l = 14.0 (dan 14.4)—15.8 (dan 15.1) mm. De L B-index is L B = 61 (dan 64)—69. Van 6 bonen is de breedte b = 9.1—9.4 mm, de lengte is l = 13.7 (14.0)—15.0 (14.2) mm. L B = 63—67. Er zijn ten slotte 12 bonen met de breedte, b = 8.1—8.9 mm, de lengte is, l = 12.8—15.2 (14.2) mm. L B = 58—65.

De bonenopbrengst van pl. 224 bevat een minder groot aantal bonen met een grote breedte dan die van pl. 222 en 223. Er is slechts één boon met een zeer grote lengte en er zijn er vrij veel met een hoge L B-index. De uitgangsboon van pl. 292 heeft ook een

iets kleinere lengte. is echter de laatste boon van de peul. De bonenopbrengst is minder duidelijk phaenotypisch $L_1 L_2 B_1 b_2$ dan de bonenopbrengsten van pl. 222 en pl. 223; de uitgangsboon van pl. 292 voor pl. 224 heeft dus ook het genotype $L_1 L_2$ in mindere mate in de homozygote vorm.

De bonenopbrengsten van de pl. 222, 223 en 224 zijn gegroeid uit de 3 bonen van een peul met 3 bonen van pl. 292, die uitzonderlijke bonen van deze bonenopbrengst zijn. Ze hebben een grote breedte en een zeer grote lengte (blz. 668). De bonenopbrengst van de 3 planten 222—224 hebben alle het phaenotype $L_1 L_2 B_1 b_2$ in overeenstemming met de uitzonderlijke afmetingen (lengte en breedte) van de uitgangsbonen. Deze 3 uitgangsbonen hebben het genotype $L_1 L_2 B_1 b_2$.

Pl. 293, F_4-1936. De uitgangsboon is van pl. 68, F_3-1935 (tab. 6). Van de 2de boon van de peul komen de lengte en de breedte zeer met die van de uitgangsboon overeen; ze zijn zeer groot, de L B-index is hoog. Van de 23 bonen van de bonenopbrengst van pl. 293 hebben 3 bonen de breedte, b = 9.4—9.6 mm, de lengte is l = 13.8—14.4 mm; L B = 67—69. Er zijn 13 bonen met de breedte, b = 8.6—9.0 mm, de lengte is 13.8—14.4 mm; L B = 65(67)—73 (70). Van 7 bonen ten slotte is de breedte, b = 8.0—8.5 mm en de lengte, l = 11.6—12.5 mm; L B = 67—70. De bonenopbrengst heeft een gelijkmatige samenstelling. Er staat bij aangetekend: „zeer matig, vrij veel schimmel". De plant is dus misschien slecht gegroeid, de bonen achtergebleven in de groei. De bonenopbrengst heeft phaenotypisch de form. $L_1 l_2 B_1 b_2$. De uitgangsboon van pl. 68, 1935 heeft de lengte l = 15.6 mm, d.i. net boven de grenswaarde voor zeer grote lengte. De L B-index is hoog (L B = 66). We kunnen aannemen, dat de uitgangsboon een plusvariant is van bonen met het genotype $L_1 l_2 B_1 b_2$. Van pl. 293, F_4-1936 zijn in 1937 2 bonen voortgekweekt. Het zijn 2 bonen van een peul met 3 bonen, de grootste en zwaarste bonen van de bonenopbrengst van pl. 293.

Pl. 225. F_5-1937 (tab. 6). Er zijn in de bonenopbrengst van 29 bonen van pl. 225 13 bonen met de breedte b = 9.1—9.4 mm, l = 13.2(13.5)—15.1(14.6) mm. L B = 62—69 (daarvan 5 met L B = 62—65). Van 8 bonen is de breedte, b = 8.6—9.0 mm en de lengte, l = 12.5—15.2(13.8) mm. De L B-index is L B = 59(62)—70; er zijn daarbij 4 bonen met L B = 59—65. Van 8 bonen ten slotte is b = 7.6(8.0)—8.5 mm en l = 12.0—13.0 mm. L B = 62—70 (daarvan is van 6 bonen L B = 62—65). Er zijn in de bonenopbrengst van pl. 225, 16 bonen met de L B-index, L B = 59(dan 62)—65, d.i. met een nog al lage L B-index. Er zijn ook vrij veel bonen met een kleine lengte (l = 13.0 en kleiner). We hebben hier niet met een bonenopbrengst te doen, die duidelijk voldoet aan de form. $L_1 L_2 B_1 b_2$, noch aan de form. $L_1 l_2 B_1 b_2$.

Pl. 226. De uitgangsboon van pl. 393 voor pl. 226 is de 2de boon van dezelfde peul met 3 bonen, als waarvan die voor pl. 225 genomen is; de lengte is iets kleiner (tab. 6). Van de 3de boon van de peul komen lengte en breedte zeer overeen met die van de uitgangsboon voor pl. 226. In de bonenopbrengst van 29 bonen van pl. 226 zijn 3 bonen met de breedte, b = 9.6—9.8 mm; de lengte is, l = 13.4—14.2 mm. De ind. L B = 69—72. Er zijn 11 bonen, waarvan de breedte, b = 9.1—9.4 mm, de lengte is l = 13.3—14.2 mm. De L B-index is L B = 64—69; daarvan is van 2 bonen L B = 64 en = 65. Van 12 bonen is de breedte, b = 8.6—9.0 mm en de lengte, l = 12.6—14.4 (dan 13.8) mm. De L B-index is L B = 63—70; daarvan is van 5 bonen L B = 63—65. Er zijn ten slotte 3 bonen met de breedte, b = 8.1—8.4 mm en l = 11.1—13.2(11.4) mm. De L B-index is L B = 64, 71 en 75.

In de bonenopbrengst van pl. 226 zijn slechts 3 bonen met een kleine breedte en er is geen enkele boon met een index niet hoger dan 65. Het phaenotype van de bonenopbrengst van pl. 226 is voor de lengte en de breedte $L_1 l_2 B_1 b_2$. We nemen op grond daarvan aan, dat het genotype van de uitgangsboon van pl. 293 voor pl. 226 eveneens $L_1 l_2 B_1 b_2$ is.

Pl. 294, F_4-1936. De uitgangsboon is van pl. 68 (tab. 6) en van een peul met 4 bonen, die zeer met elkaar overeenkomen. De formule van de uitgangsboon is phaenotypisch kenmerkend $L_1 l_2 B_1 b_2$. De bonenopbrengst van pl. 294 is klein; n = 14, er staat bij aangetekend: „matig en groenig". Er zijn 4 bonen met de breedte, b = 9.6—10.3 mm,

$l = 13.5—14.2$ mm; $LB = 68—74$. Er zijn 10 bonen met $b = 8.7—9.5$ mm; $l = 12.0(12.5)$ —14.2 mm; $LB = 66—73$. Alle 14 bonen hebben de form. $L_1 l_2 B_1 b_2$. Van pl. 294 is één boon in 1937 voortgekweekt. Ze leverde pl. 227, F_5-1937, (Tab. 6). De uitgangsboon is van een peul met 3 onderling overeenkomende bonen. Ze heeft de grootste breedte van de bonenopbrengst. De formule is $L_1 l_2 B_1 b_2$. De bonenopbrengst van 30 bonen van pl. 227 bevat één boon met de breedte, $b = 9.6$ mm; de lengte is $l = 14.6$ mm; $LB = 66$. Er zijn 16 bonen met de breedte $b = 8.6—9.4$ mm; de lengte van deze bonen is, $l = 12.4—14.1$ mm; $LB = 63—71$. Van 2 bonen is LB, resp. 63 en 65. Er zijn 13 bonen met $b = 7.4—8.4$ mm, (daarvan is van 3 bonen, $b = 7.4—7.6$ mm, ze zijn de laatste in de rij van de peul; de lengte is, $l = 11.5$, 11.6 en 11.8 mm; $LB = 63$, 64 en 66). De lengte is 11.4—13.7(13.3) mm. $LB = 64(66)—71$.

In de bonenopbrengst van pl. 227 zijn 16 bonen met een grote breedte en er is geen boon met een zeer grote lengte. Er zijn 5 bonen met een LB-index, die niet hoger is dan 65. De formule van de bonenopbrengst is $L_1 l_2 B_1 b_2$; ze is daarvan een goed voorbeeld. We nemen aan, dat de uitgangsboon voor de lengte en de breedte het genotype $L_1 l_2 B_1 b_2$ heeft.

De bonenopbrengst van pl. 68, F_3-1935 met haar descendentie in de pl. 291—294, F_4-1936 en in de pl. 221—227, F_5-1937 is een goed voorbeeld, weliswaar in mindere mate dan die van pl. 58, F_3-1935 van de erfelijkheid van bonen met de formule $L_1 l_2 B_1 b_2$ voor de lengte en de breedte.

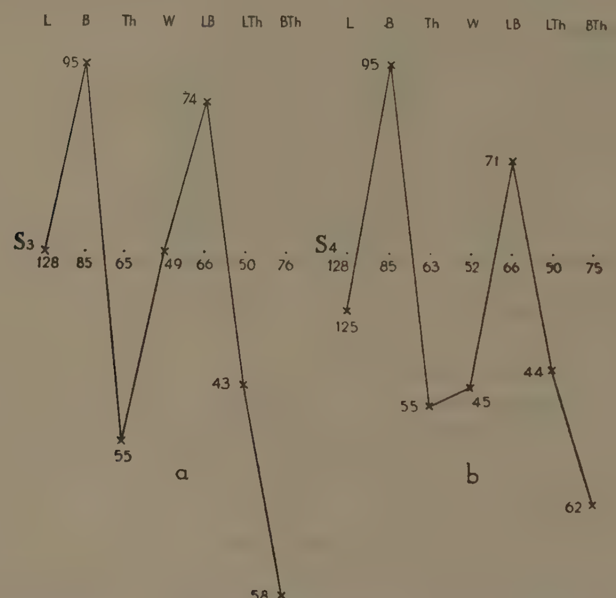


Fig. 6. *a.* Characterogram of 6 p 2 b, initial bean of pl. 259, F_4-1936 for pl. 207, F_5-1937
b. Idem of the averages of pl. 207, F_5-1937.

Uit de meegedeelde resultaten over de pl. 73, 93, 58 en 68 van ons materiaal van F_3-1935, blijkt de erfelijkheid van bonen met de formule $L_1 L_2 B_1 b_2$ en van die met de formule $L_1 l_2 B_1 b_2$, dat is van bonen met een grote breedte en een zeer grote lengte en van bonen eveneens met een grote breedte, doch een niet zeer grote lengte. We beschouwen deze bonen als uitsplitsingen, dat is resultaat van splitsende erfelijkheid.

Het voorkomen van bonen met een grote breedte en een zeer grote lengte en daarnaast van bonen met eveneens een grote breedte en een niet zeer grote lengte, waarbij deze eigenschappen erfelijk zijn, hebben we afgeleid uit onze hypothese over de erfelijkheid van de afmetingen bij kruisingsproeven en we vinden deze verwachting in ons materiaal van de F_3-generatie 1935, bevestigd. Het resultaat is belangrijk voor onze hypothese, waarvan we de betekenis vroeger uiteenzetten (Proceed. 50, 1947).

Onze hypothese gaat uit van de mendelistische erfelijkheid en het verrichte onderzoek brengt nieuwe feiten. Deze blijven nodig bij de ingewikkeldheid van het verschijnsel erfelijkheid en zijn elementaire betekenis.

We vinden splitsende erfelijkheid en vatten onze resultaten op als een bevestiging van de theorieën van MENDEL en van MORGAN [1]) over de erfelijkheid. Daarbij aanvaarden we de consequentie van de chromosomentheorie van MORGAN, dat het nodig is, om aan te nemen, dat de chromosomen van de celkern de plaatsen beschikbaar hebben voor de genen van alle variëteiten, waarmee een plant of dier kruising kan aangaan (allelomorphisme, BATESON). Voor de mens wil dit zeggen, dat de chromosomen van de celkern bij de normale mens de plaatsen bevatten voor de genen van alle anomalieën, die bij de mens voorkomen. Op deze consequentie is ook door anderen gewezen (VON VERSCHUER).

Waar de theorie van de erfelijkheid van MENDEL en MORGAN voor een bepaald gebied, dat der kruisingen, opgesteld is, kan de vraag gesteld worden, of ze als theorie van het gen nog wijder geldigheid heeft. Het optreden van mutaties, dat zijn erfelijke variaties, wordt teruggebracht tot mutaties van het gen en evolutie wordt verklaard door genen-mutatie. H. J. MULLER [2]) beschouwt het gen als de grondslag van het leven en het begin van de evolutie. Uit het gen ontstaat, als erfelijke variatie het gemuteerde gen. Op deze wijze kunnen we de verscheidenheid van levende wezens verstaan, de toename van het aantal genen in hetzelfde organisme wordt er niet door verklaard. Dit is een moeilijk punt voor de evolutietheorie door mutabiliteit. De lineaire rangschikking van de genen in de kernlissen bepaalt, dat het nieuwe, gemuteerde gen komt te liggen op de plaats, de locus, van het muterende gen. Daarmee stijgt niet het aantal genen. MULLER wijst er op, dat het verschijnsel van duplicatie van chromosomen-materiaal in experimenten met radiumbestraling, waardoor een stukje van een chromosoom losraakt en gebracht wordt naar een andere plaats, tot de toename van het aantal genen in de kernlissen in de loop der evolutie kan hebben geleid. Vermeerdering van het aantal genen, zegt MULLER, teweeggebracht door de duplicatie van kleine delen van chromosomen, moet worden opgevat als één van de voornaamste processen, waar-

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[1]) TH. MORGAN. The relation of Genetics to Medicine. Nobel lecture, Stockholm, 1934. The Scientific Monthly, **41**, 5 (1935). The Theory of the Gene, 1926. The Scientific Basis of Evolution, 1932.

[2]) H. J. MULLER. The Gene as the Basis of Life. Internat. Congr. of Plant. Sc. 1926; in Proceed. of the Congress **1**, 897 (1929).

op de evolutie berust, tezamen met de mutaties in de individuele genen [1]). Na de duplicatie kunnen zich genenmutaties voordoen, waardoor de genen in het nieuwe deel van de duplicatie kunnen gaan verschillen van die van het oorspronkelijke deel. Het chromosomen-materiaal van het oorspronkelijke gen, als grondslag van leven, zou dus ook door duplicatie en mutatie tot vermeerdering van genen in de celkern hebben geleid. Aan een verschijnsel, dat der duplicatie, van het gebied der afwijkingen in het chromosomen-gedrag bij de kerndeling, wordt hier grote betekenis toegekend in het tot stand komen van een natuurlijk verschijnsel, normaal verschijnsel in de natuur, de evolutie. Door verder onderzoek en experiment, door het verzamelen van feiten, zal blijken, of de theorie van MENDEL en MORGAN, die geldt voor het gebied der kruisingen, wijder geldigheid heeft.

Erfelijkheid is een veelomvattende en elementaire eigenschap van het leven. Ze is het conserverende vermogen van het levende organisme.

Erfelijkheid, variabiliteit en aanpassing maken het zijn van het levende organisme in de natuur mogelijk.

De evolutie is een algemeen verschijnsel in de natuur, de levenloze en de levende. De evolutie van planten en dieren is resultaat van zich wijzigend milieu en zich aanpassende aanleg. Het vermogen tot aanpassing houdt het vermogen tot evolutie in.

Summary

According to the hypothesis of the report in the Proc. Vol. 50, p. 798, 1947, the formulas started from are:

$$L_1 L_2 \; B_1 b_2 \; th_1 th_2$$ for beans of the I-line and

$$l_1 l_2 \; b_1 b_2 \; Th_1 th_2$$ for beans of the II-line.

For the heredity of the differences of the dimensions 6 pairs of genes were assumed. The size-increase through one gene is equally great relatively for all the dimensions. One gene for the length, one for the breadth and one for the thickness are always active at the same time. In order to explain the much greater difference of the lengths than of the breadths and the thicknesses it is assumed that beans of the I-line have 6 genes for the length in the homozygous, dominant form (as L L) whereas they have only 3 genes for the breadth in the homozygous, dominant form (as B B). Beans of the II-line have only 3 genes for the thickness in the homozygous, dominant form (as Th Th). Of the 6 genes for the dimensions therefore the genes 4—6 for the breadth and for the thickness only occur in the homozygous, recessive form (as b b and th th). For this reason we write the formulas as we did above. It follows from these formulas that we have to deal with a cross according to the tetrahybrid scheme.

This communication busies itself with the crossproducts $L_1 L_2 \; B_1 b_2$ and

[1]) H. J. MULLER. The Production of Mutations Nobel Price Lecture. Stockholm, Dec. 12, 1946. In Journ. Hered. Vol. 38, p. 259, Sept. 1947.

$L_1 l_2 B_1 b_2$ of the material F_3-1935, which will appear according to our hypothesis. They are resp. beans with a great breadth and a very great length and beans with a great breadth and a not very great length. The limit of the lengths is 15.5 millimeters.

Tab. 3 shows that among the beans with a great breadth, $b = 11.3$—10.1 mm, of the F_3-generation 1935, there are several with a not very great length $l = 15.5$—13.6 mm. Among beans with such a great breadth of the I-line of 1935, we only find one or two with a not very great length. Among the F_3-material there are beans with such a small length as it is not found among the beans of the I-line.

This different finding of beans with a great breadth of the F_3-generation and of the I-line can be explained from the different hereditary composition of these beans. The formula for the length and the breadth is the same for all the beans of the I-line nl. $L_1 L_2 B_1 b_2$. Among the beans with a great breadth of the F_3-material there are some with the form. $L_1 L_2 B_1 b_2$ as of the I-line, but also some with the form. $L_1 l_2 B_1 b_2$.

We see from tab. 4 that beans with the form. $L_1 l_2 B_1 b_2$ for the length and the breadth have a great absolute breadth and a high L B-index.

We see from tab. 5 that there are a few F_3-beanyields with a great average breadth and a not very great average length. They answer to the form. $L_1 l_2 B_1 b_2$.

Of beanyields with a great average breadth of the I-line, the average length is always very great too.

The heredity of beans of F_3-1935, with the formulas of the length and the breadth resp: $L_1 L_2 B_1 b_2$ and $L_1 l_2 B_1 b_2$ we investigated for some beanyields of tab. 5 (tab. 6).

Pl. 73. The beanyields of pl. 73 with a great average breadth has the greatest average length of the beanyields of the F_3-generation of 1935. The initial bean for pl. 73 is of pl. 81, F_2-1934 and has likewise a very great length (the limit) and a great breadth (tab. 6). The beanyield of pl. 73 (tab. 7) contains many beans with a great breadth and a very great length (form. $L_1 L_2 B_1 b_2$). There is not one with a great breadth and a not very great length, (form. $L_1 l_2 B_1 b_2$). There is a correspondence with beanyields of the I-line of 1935 (tab. 7). According to the analysis (p. 580) the beanyield of pl. 309, F_4-1936 (tab. 6) contains many beans with the formula $L_1 L_2 B_1 b_2$ of the length and the breadth. This composition points to the heredity of this formula of the initial bean of pl. 73 for pl. 309.

The initial bean of pl. 309 for pl. 239, F_5-1937 (tab. 6) is the bean with the greatest breadth and the greatest length of the beanyield of pl. 309. The composition of the beanyield of pl. 239 (p. 581) answers to the form. $L_1 L_2 B_1 b_2$ for the length and the breadth. The initial bean of pl. 309, F_4-1936 for pl. 239 has the form. $L_1 L_2 B_1 b_2$ in a homozygous or nearly homozygous form.

From the composition of the beanyields of pls. 151—153 (p. 582) it appears that the initial beans of pl. 309 for these plants do not have

the form. $L_1 L_2 B_1 b_2$ in the homozygous form; in this they lay behind that for pl. 239.

The data for 4 generations of the descendency and of the ascendency (1934—1937) of the beanyields of pl. 75, F_3-1935, show that of these beanyields with the formula $L_1 L_2 B_1 b_2$ for the length and the breadth the characteristics great breadth and very great length are hereditary here (fig. 1).

Pl. 93. The beanyield of pl. 93 has the smallest average length of the beanyields with a great average breadth of the F_3-generation of 1935. Beanyields with such a small average length do not occur among the beanyields with a great average breadth of the II-line (tab. 5).

The initial bean of pl. 82 for pl. 93 (tab. 6) has a moderately great breadth and a very great length. The remaining 5 beans of the pod of the initial bean all have a smaller length. The beanyield of pl. 93 (tab. 7) contains a great many beans with a great breadth; among these there is not one with a very great length.

The beanyields of the pls. 352, 353 and 354, F_4-1936 of initial beans of pl. 93, F_3-1935, show an affinity with beanyields with the form. $L_1 l_2 B_1 b_2$ but they are no clear instances of it. The beanyields of 1936 are not entirely comparable with those of 1935, because the crop of 1936 was less good (tab. 2). The descendency and the ascendency in 3 generations of pl. 93 point to the facts that this beanyield is composite, that the initial bean of pl. 82 for pl. 93 has the form. $L_1 l_2 B_1 b_2$ for the length and the breadth only in a heterozygous form.

Pl. 58, F_3-1935 is a 2nd instance of a beanyield with a great average breadth and a not very great average length of which the data about the heredity have been discussed here (tabs. 5—7, figs. 4—6). The beanyield of pl. 58 contains a great many beans with a great breadth and a not very great length. It answers quite well to the requirements for a beanyield with the form. $L_1 l_2 B_1 b_2$. Also the phaenotype of the initial bean of pl. 70 for pl. 58 has this formula.

Pls. 259 and 260, F_4-1936, (tab. 6) have grown from initial beans of pl. 58, F_3-1935. A few full grown beans of the beanyield of pl. 259 have a great breadth and a not very great length. The other beans are small probably in consequence of stunted growth. The LB-indices of all the beans are high; the genotype can therefore be $L_1 l_2 B_1 b_2$.

Pls. 206 and 207, F_5-1937, (tab. 6) have grown from 2 initial beans with the greatest breadths of pl. 259, F_4-1936. According to the analysis of the beanyield of pl. 206 (p. 585) it answers quite well to the phaenotype $L_1 l_2 B_1 b_2$ and the initial bean for this reason to the genotype $L_1 l_2 B_1 b_2$. This applies in an even greater measure to pl. 207. Of this too we assume that the initial bean has the form. $L_1 l_2 B_1 b_2$ in a homozygous or nearly homozygous form.

From a 2nd initial bean of pl. 58, pl. 260, F_4-1936 has grown and from an initial bean of this pl. 208, F_5-1937 (tab. 6). According to the analysis

(p. 585) the beanyield of pl. 260 answers quite well to the phaenotype $L_1 l_2 B_1 b_2$. We therefore assume that the initial bean of pl. 58 is a non-hereditary variation having the genotype $L_1 l_2 B_1 b_2$. According to the analysis of pl. 208, F_5-1937 (p. 586) this beanyield has the phaenotype $L_1 l_2 B_1 b_2$ fairly well. The beanyield of pl. 260 and its initial bean of pl. 58 answer to the form. $L_1 l_2 B_1 b_2$ in a less homozygous form than the beanyield of pl. 259 and its initial bean of pl. 58. In the beanyield of pl. 58, F_3-1935, its initial bean of pl. 70, F_2-1934, the beanyield of pl. 259, F_4-1936 and its initial bean of pl. 58 and the beanyield of pl. 207, F_5-1937 and its initial bean of pl. 259 we have a good instance of the heredity of beans with the formula $L_1 l_2 B_1 b_2$ for the length and the breadth in 4 generations (tab. 6 and figs. 3—6).

Pl. 68, F_3-1935 is the 3rd instance that is discussed of a beanyield with the form. $L_1 l_2 B_1 b_2$ for the average length and breadth (tab. 6). From the composition of the beanyield we deduct the fact that the genotype of the initial bean of pl. 76, F_2-1934 is to a high degree homozygous for the form. $L_1 l_2 B_1 b_2$. Four initial beans of pl. 68 gave the pls. 291—294 as F_4-1936. According to the analysis of these beanyields (ps. 667—669) their phaenotypes are $L_1 l_2 B_1 b_2$. Of the initial beans of pl. 68 for the pls. 291 and 293 we assume that they are non-hereditary variations of beans with the form. $L_1 l_2 B_1 b_2$ in a predominantly homozygous form.

Of the pls. 292—294, F_4-1936 some beans were taken for a new generation; they gave pls. 221—227, F_5-1937. Remarkable are pls. 222—224, grown from the 3 beans of a pod of pl. 292, all of them exceptionally large and heavy beans, the only pod of pl. 292 with beans of this kind. The formula of these initial beans is $L_1 L_2 B_1 b_2$ (tab. 6). The composition of the beanyields (ps. 667—668) shows that all three of them have the phaenotype $L_1 L_2 B_1 b_2$ in correspondence with the very great length and great breadth of the initial bean for which we therefore assume the genotype $L_1 L_2 B_1 b_2$. These 3 beans of one pod of pl. 292 are a good instance of segregates in a beanyield which for the rest answers to the composition $L_1 l_2 B_1 b_2$.

From 2 initial beans of pl. 293, F_4-1936, grew the pls. 225 and 226, F_5-1937 and from an initial bean of pl. 294 grew pl. 227, F_5-1937 (tab. 6).

The beanyield of pl. 68, F_3-1935 with its ascendency in the pls. 291—294, F_4-1936 and in the pls. 221—227, F_5-1937 is a good instance, to a lesser degree it is true, than that of pl. 58, F_3-1935, of the heredity of beans with the formula $L_1 l_2 B_1 b_2$ for the length and the breadth.

From the results given of the pls. 73. 93, 58 and 68 of our material of F_3-1935 appears the heredity of beans with the form. $L_1 L_2 B_1 b_2$ and of those with the form. $L_1 l_2 B_1 b_2$, that is of beans with a great breadth and a very great length and of beans with also a great breadth but with a not very great length.

The report ends with a few remarks on the theory of the gene.

Petrology. — *Petrology of the Mt.-Aigoual area in the southeastern Cevennes, France.* By R. C. HEIM. (Communicated by Prof. J. H. F. UMBGROVE.)

(Communicated at the meeting of May 28, 1949.)

1. *Summary and conclusions.*

A petrographical description is given of the Mt.-Aigoual pluton, its associated dikes and the country rock.

Particular features of phenocrysts warrant the conclusion that the process of formation of the plutonic rocks is not to be vizualized as a simple crystallization of a fluid at rest after injection, but that a long proces of crystallization and recrystallization has been going on before and during the emplacement of the rock-mass.

2. *Introduction.*

On the geological map of France, scale 1 : 80.000, sheet Alais, the northern part of the Mt.-Aigoual pluton is shown. In the course of investigations carried out in the summer of 1947 and 1948, detailed mapping proved that the pluton has a much smaller outcrop than shown on the map; the northern border being in reality about two km more to the south. The area in excess is occupied by slates with a great number of dikes, mostly of granodioritic composition.

A short petrographic description of the different kind of rocks encountered in this region (fig. 1) is given here as a supplement to the tectonic description by DE WAARD (1949-*a*, *b*).

3. *Country rock.*

The age of the slates has not been established for certain, but probably they are of Cambrian age (DEMAY, 1931-*a*). They form part of a great unit of monoclinal slates, dipping towards the NE, covering nearly half of the sheet Alais. Folding is considered to be of Carboniferous age — post-Visean and pre-Westphalian; the latter formation not being affected by this folding in the adjacent basin of Alais (DEMAY, 1931-*b*).

That the emplacement of the pluton took place after this folding is shown by the fact that the folded slates are truncated by the pluton.

North of the outcrop of the pluton, hills are covered by flat lying Mesozoic strata, whose base is formed by a coarse arkose consisting of unaltered elements of the plutonic rocks — abundant phenocrysts of orthoclase are very conspicuous — with an admixture of material derived from the slates. Near this basal conglomerate the underlying plutonic rocks and slate are of a brick red colour which, by some authors, is regarded as

indicative of desert [1]) conditions. Obviously no tectonic and magmatic activity occured after the deposition of the arkose, which can be dated as Triassic.

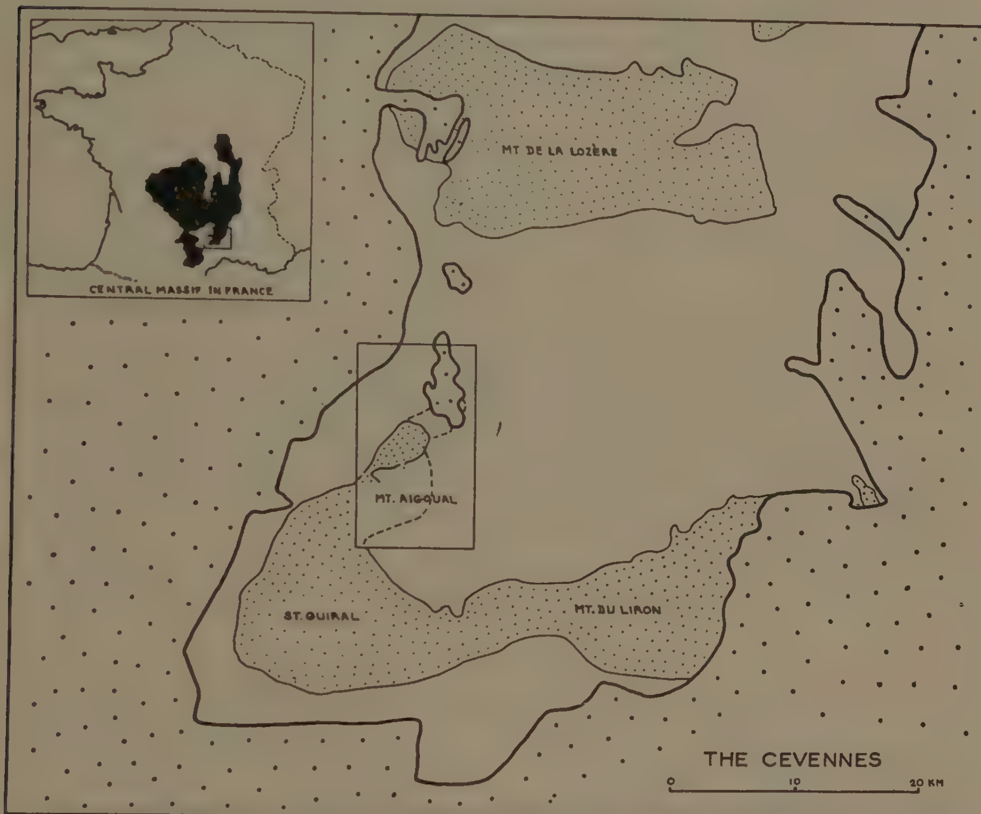

Fig. 1. Geographical position of the investigated area. Slates are white, granite massifs dotted, Mesozoic strata coarsely dotted. From: DE WAARD, Tectonics of the Mt. Aigonal pluton in the northeastern Cevennes, France. Part. I. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 52.

The slates are for the greater part phyllitic, consisting of granular quartz, muscovite and biotite. Sometimes the only mica is biotite. The quartz content changes locally; in some places we find layers of compact quartzite between the phyllites.

In nearly all our samples quartz makes up more than half of the rock. Generally part of the biotite has been converted into chlorite; in some cases we find nothing but chlorite and muscovite. Occasionally some of the chlorite is developed as porphyroblasts, usually with polysynthetic twinning (clinochlor), sometimes showing helicitic structure. No spacial relation could be discovered between chloritization and intrusion of the pluton. The biotite has always inclusions of zircon, rutile or apatite.

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[1]) See, however, P. D. KAYNINE, The origin of red beds, Transact., New York Academy of Sciences, ser. II, vol. 71, 1949, pp. 60—68.

In the direct vicinity of the pluton or of a dike the slate often contains a small quantity of orthoclase or acid plagioclase; in this case a slight baueritization of the biotite can be observed. In one sample of feldspathic slate all the quartz has been replaced by orthoclase, the feldspar still showing the original texture of the quartz. Occasionally minerals like cordierite, sillimanite and granular andalusite occur in the slate within a distance of 100 m from its contact with the pluton; they do not form great porphyroblasts. The cordierite has been partly transformed into pinite.

In the northern part of the mapped region, however, where dikes are rare, many slates were encountered with relatively great porphyroblasts (up to 0,5 cm) of pink garnet and cyanite (pleochroism light-blue to colourless). The great distance from the outcrop of the pluton and the scarcity of dikes allmost exclude the possibility of contactmetamorphism.

Most of our samples contain some tourmaline, a common occurence in slates. Some samples contain minute flakes of graphite.

Always present, but in changing quantities, are lenticular intercalations of quartz, having a size of a few cm to several dm. They were formed before the intrusion of the pluton, for they were affected by the folding, which was anterior to the intrusion.

4. *Pluton.*

The mineralogical composition [2]) of two samples from the pluton was:

	I.	II.
Plagioclase	30 (38 % an.)	37 (30 % an.)
orthoclase	18	29
quartz	28	19
biotite	22	14
hornblende	2	1

The grey rock is medium-grained but contains phenocrysts of white orthoclase with a size up to 15 cm. The amount of phenocrysts changes locally. The normal facies contains plagioclase in excess of orthoclase, therefor the rock can be called a granodiorite (according to LINDGREN (1900) the amount of plagioclase in a granodiorite ought to be at least double the amount of orthoclase, and the rock of the Mt.-Aigoual should be called a quartz-monzonite). MICHEL–LÉVY (1939) gave a chemical analysis of "le granite du Mt.-Aigoual". His statement that in this rock the amount of orthoclase is much larger than that of the plagioclase is not in accordance with our observations, nor with the norm following from his chemical analysis.

The phenocrysts, which are often perthetic, have numerous inclusions of quartz, plagioclase and biotite, which occur in their very center as well as in more peripherous parts, as was found by grinding down some of the phenocrysts. The crystallization of the phenocrysts and of the other

[2]) All compositions given here are volumetrical.

minerals must have been well on its way before the intrusion took place, because the phenocrysts show distinct flow-orientation (DE WAARD), 1949-*a*, *b*).

Both generations of the orthoclase have an optical angle of about 57°. In the groundmass micrographic intergrowth of quartz and orthoclase, and antiperthite have been observed. The quartz shows undulatory extinction. The biotite is brown, and shows numerous inclusions of apatite and zircon. The hornblende is light-green, often twinned. Near the border of the pluton the biotite is chloritisized, or baueritisized, containing saginite, and lenses of quartz and calcite; in the feldspar sericite is formed. At the contact are some veins of pegmatite and along a zone of a few metres several xenoliths of feldspathic slates occur.

5. *Dikes.*

Most of the dikes consist of granodiorite-porphyry; some more acid dikes were found: granite-porphyry, aplite, quartz; and more basic dikes of tonalite-porphyry, diorite-porphyry and kersantite.

All these dikes cut the slates, and dikes of tonalite-porphyry, kersantite and granite-porphyry have been observed in the pluton. The dikes have straight and parallel boundaries, often interrupted by irregular protuberances. The contact with the slates is always sharp. Their width varies between a few decimetres and several metres.

Granodiorite-porphyry.

The mineralogical composition of this rock closely resembles the granodiorite of the pluton, but apart from orthoclase, there are phenocrysts of quartz and sometimes of hornblende (all more or less idiomorphic). The groundmass is finer grained (but always holocrystalline) and hydrothermal alteration is more intense. As a rule the phenocrysts are orientated parallel to the walls of the dike. In both generations the plagioclase contains 30—40 % anorthite. In the groundmass quartz and orthoclase are often micrographically intergrown. The biotite has numerous inclusions of apatite and zircon, and often of saginite. For the greater part the feldspars have been seriticized. Biotite and hornblende have been partly altered into chlorite; many biotites have been bleached and contain lenses of quartz, epidote and calcite. Myrmekite is common. Quartz phenocrysts show undulatory extinction and have been broken; in some cases the fragments show homogeneous extinction. Relatively much apatite was found.

Granite-porphyry.

This is a yellowish-white rock with tiny quartz phenocrysts, no orthoclase phenocrysts and a very dense but holocrystalline groundmass. The exact mineralogical composition could not be determined; apparently it was intermediate between granite and aplite, with little biotite, and orthoclase in excess of plagioclase (5 % an.). This rock has been found forming the margin of some of the granodioritic dikes and at its contact it penetrates

the slate by numerous little offshoots, in contrast with the granodiorite-porphyry.

Aplite.

In the pluton are some small dikes of aplite, a few decimetres thick, consisting allmost entirely of orthoclase and quartz. They abound locally at the border of the outcrop, cutting pluton, country rock and granodioritic dikes.

Quartz.

Some straight dikes of nearly pure quartz were observed, having a width of a few decimetres.

Tonalite-porphyry.

Some granodiorite-porphyries have a darker groundmass than the normal granodiorite-porphyry. They contain less orthoclase phenocrysts and often have a microfelsitic basis (ZIRKEL, 1893). They form a transition to the tonalite-porphyry. Both rocks have been seen cutting granodioritic dikes and country rock alike.

The tonalite-porphyry is a fine-grained, dark-grey rock with scarce phenocrysts of quartz, plagioclase and biopyroboles. The groundmass, often trachytic, contains the same minerals; the plagioclase is neatly idiomorphic. One sample showed the following composition:

plagioclase	42	(35 % an.)
quartz	6	
hornblende	30	
biotite	22	

Occasionally chlorite and sericite occurs.

The quartz phenocrysts are usually surrounded by a rim of hornblende crystallites.

Diorite-porphyry.

This rock strongly resembles the tonalite-porphyry, but it contains less than 5 % quartz, which is only interstitial.

In one dike a melanocratic rock was found, which resembled an odinite. It is a greenish-grey rock with a dense, doleritic groundmass of plagioclase-laths, hornblende and biotite, and a little interstitial quartz. It has phenocrysts of plagioclase, uralitisized augite and a few phenocrysts of quartz; the latter are surrounded by a rim of hornblende crystallites. The mineralogical composition of this odinite-like rock is:

plagioclase	42	(52 % an.)
quartz	3	
hornblende	33	
augite (uralite)	12	
biotite	10	

Kersantite.

The fine-grained, porous groundmass consists of acid plagioclase, a little light-green hornblende, quartz and some orthoclase, in micrographic intergrowth with quartz. The rock is generally much altered; kaoline, sericite, chlorite and calcite were found. One sample has the following composition:

plagioclase	51
orthoclase	4
quartz	5
biotite	23
chlorite	17

Most of the kersantite-dikes are situated at a greater distance from the outcrop of the pluton than the other dikes.

6. *The petrogenetic sequence.*

The relatively sharp contacts of the granodiorite with the country rock and the slight metamorphism of the latter suggest the granodiorite to be an intrusive body; DE WAARD (1949, *a*) found structural data leading to the same conclusion. The observations mentioned above do not exclude the possibility of the granodiorite having been generated by "granitization". This origin would perhaps afford an explanation of the inclusions in the orthoclase phenocrysts, mentioned under the heading "pluton". The flow structures shown by these phenocrysts (DE WAARD, 1949, *a*) indicate that the granodiorite was a mush of crystals during its intrusion.

All the differentiates have been observed to cross-cut pluton or granodioritic dikes; hence they are of younger age. Aplite and pegmatite intercross each other.

Aplite and pegmatite were only found in the direct vicinity of the pluton, generally forming small veins or dikes of changing width and direction, whereas the kersantites, mostly located at a far greater distance from the outcrop of the pluton, form bold, continuous dikes. If differentiation took place in a magma which was only partly fluid, as suggested above, it was probably due to the squeezing out of the mush, or — in more general terms — a relative movement of different components due to different reactions to strain. The resulting congregation of acid and basic material took place at a higher and a lower level in accordance with the difference in specific gravity. The deapseated source of the kersantites is in accordance with their large distance from the pluton.

7. *The origin of quartz phenocrysts.*

In nearly all rock-slides containing quartz phenocrysts the latter show sinusoidal outlines such as mostly are ascribed to resorption. It seems difficult, however, to find a satisfactory explanation of resorption of quartz phenocrysts by a groundmass rich in quartz. LAEMMLEIN (1930, 1933),

who ascribed the so called corrosion of quartz to unregular growth of the crystals, made the remark that resorption will tend to round off sharp edges, but will never cause holes with a narrow inlet. In our slides quartz phenocrysts have been found with rounded edges, but in these cases the former outline of the crystal is revealed by a rim of granophyre, indicating the replacement of part of the quartz by orthoclase.

The form of several other quartz phenocrysts (in slides of the rocks described under the headings "granodiorite-porphyry", "granite-porphyry" and "tonalite-porphyry") are very suggestive of a mechanical deformation of the quartz and of the formation of large quartz phenocrysts by agglomeration of several smaller units (figs 2, 3, 4, 5 and 6).

Figs. 2, 3 and 4 are photographs of a section of a quartz phenocryst. It seems (fig. 3) that the parts I and II (fig. 2) were originally two individuals; while drifting together their outlines were moulded onto each other, and a biotite crystal was caught in between. In a similar way the biotite crystal at the NW-border of the phenocryst was pushed into the quartz; plagioclase-laths of the groundmass were trapped between biotite and quartz (fig. 4). After close observation, especially of fig. 3, the phenocryst is seen to be built up of more individuals than the two already mentioned (numbered in decreasing order of distinctness 1, 2, 3 and 4).

The quartz phenocryst of fig. 5 appears in fig. 6 to be built up from at least three individuals, numbered 1, 2 and 3 in fig. 5. Between 1 and 2 again a biotite crystal has been surrounded by the quartz.

HOLMQUIST (1915) has described "corroded" quartz crystals whose bays were filled with orthoclase which also formed inclusions in the quartz. He considered both phenomena to be the result of simultaneous crystallization of quartz and feldspar. This, however, does not hold in our cases, where embayments and holes contain patches of groundmass or biotite crystals.

We therefor conclude that the quartz was in a condition which permitted a lasting deformation — though very likely an extremely slow one. Such deformations are also indicated by its very common undulatory extinction. The open, sieve-like structure with its intricate pattern of interwoven -O-Si-O-Si- chains accounts for the strong tendency of SiO_2 to remain in the glassy (molten) state and suggests the possibility of irregular connections — which is the same as to say that recrystallization is on its way, as must be the case where undulatory extinction is exhibited. At a temperature well below the melting point, this interchange of connections will occur at such a rate that finite deformation or complete recrystallization may result in the course of geological time. The open structure favours the inclusion of H_2O molecules in the presence of some water; the tendency to form a gel (opal) adds to the mobility resulting from the glassy disorder.

We thus form a picture of units of an SiO_2 phase with a behaviour intermediate between that of drops of liquid and solid crystals. Under differential movement of the rock-(magma-)mass small fragments can meet

Fig. 2. × 11.

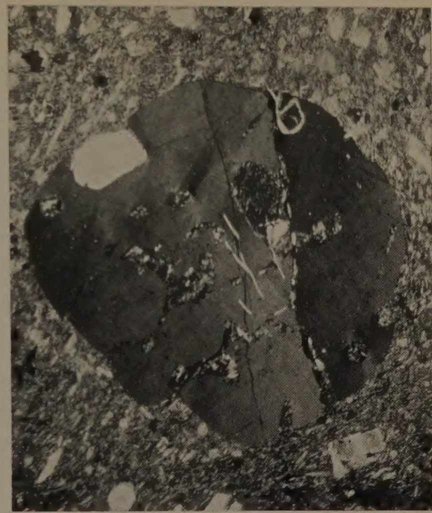

Fig. 3. × 10. Crossed nicols.

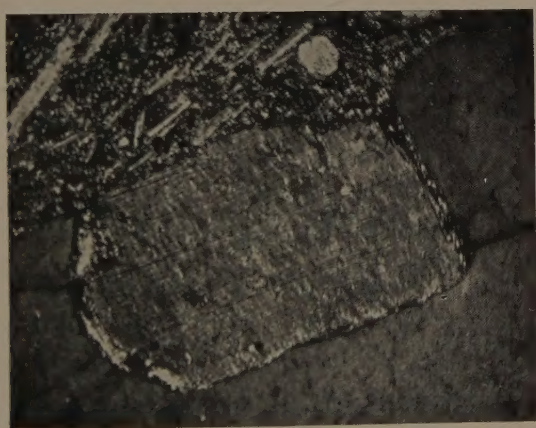

Fig. 4. × 51. Crossed nicols.

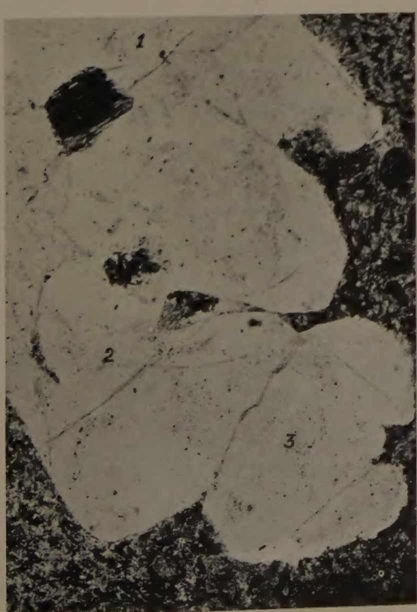

Fig. 5. × 11.

Fig. 6. × 10. Crossed nicols.

and be welded together. Recrystallization in the course of time may provide a reorientation, resulting in a crystal pattern which may run through several formerly separated units. The irregular outline of such a composite crystal thus is not a result of corrosion; its embayments mark the joints of the composing parts.

The required strong differential movement will have occured during the intrusion of the dikes but is not to be expected in a big continuous mass like a pluton, which is therefore free from quartz phenocrysts.

The quartz phenocrysts in the tonalite-porphyry and in the diorite-porphyry are usually surrounded by a rim of hornblende-crystallites. A similar feature was also noticed by HÖPFNER (1881) in a dacite, and by LARSEN in a granodiorite, which after consolidation had been partly molten by an andesite intrusion (LARSEN and SWITZER, 1939). In the latter case it is obvious that the quartz crystals, now surrounded by glass, were not in equilibrium with the glass during the melting, because the quartz was interstitial in the original granodiorite. In a region where basic and acid rocks with quartz phenocrysts occur, HOLMQUIST (1899) found, that only in the basic rocks quartz phenocrysts are traversed by strings of hornblende-crystallites. In this case the quartz phenocrysts, surrounded by the hornblende rim, must have drifted together. ROSENBUSCH (1910) states that a rim of augite crystals proves the quartz to be xenolithic. So it seems probable that in the Mt.-Aigoual region some of the quartz units of the original mush were squeezed out together with the material which later on consolidated as basic dike rock.

8. *Dikes of quartz.*

The mode of occurence of the dikes consisting of pure quartz is essentially different from that of the quartz exudations in the slates; the former are younger, because they cut the folded slates. It is possible that they are related genetically to the granodiorite, although similar dikes occur all over this part of the Central Massif.

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